Deciding language inclusion problems using quasiorders

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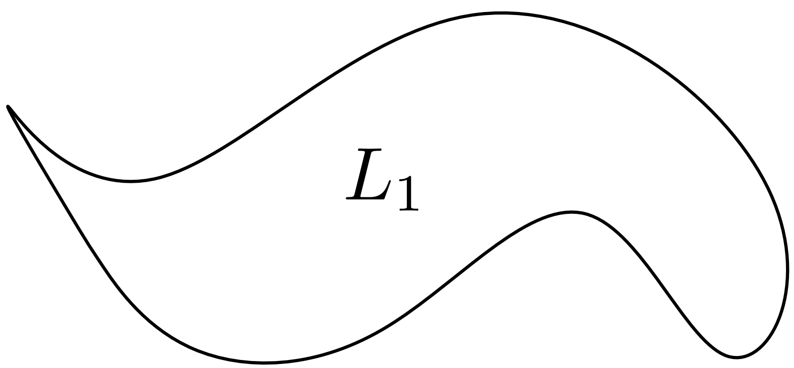
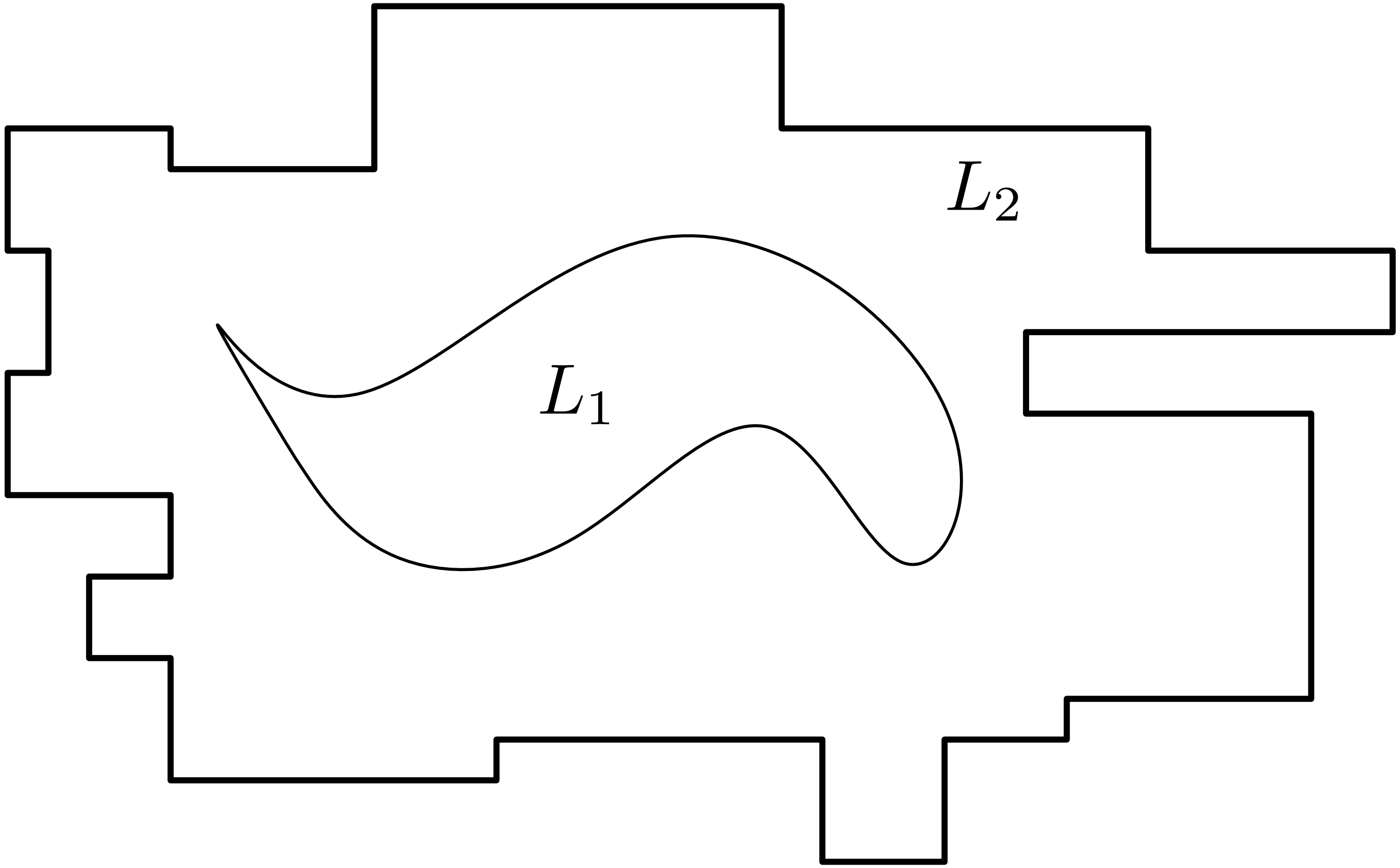
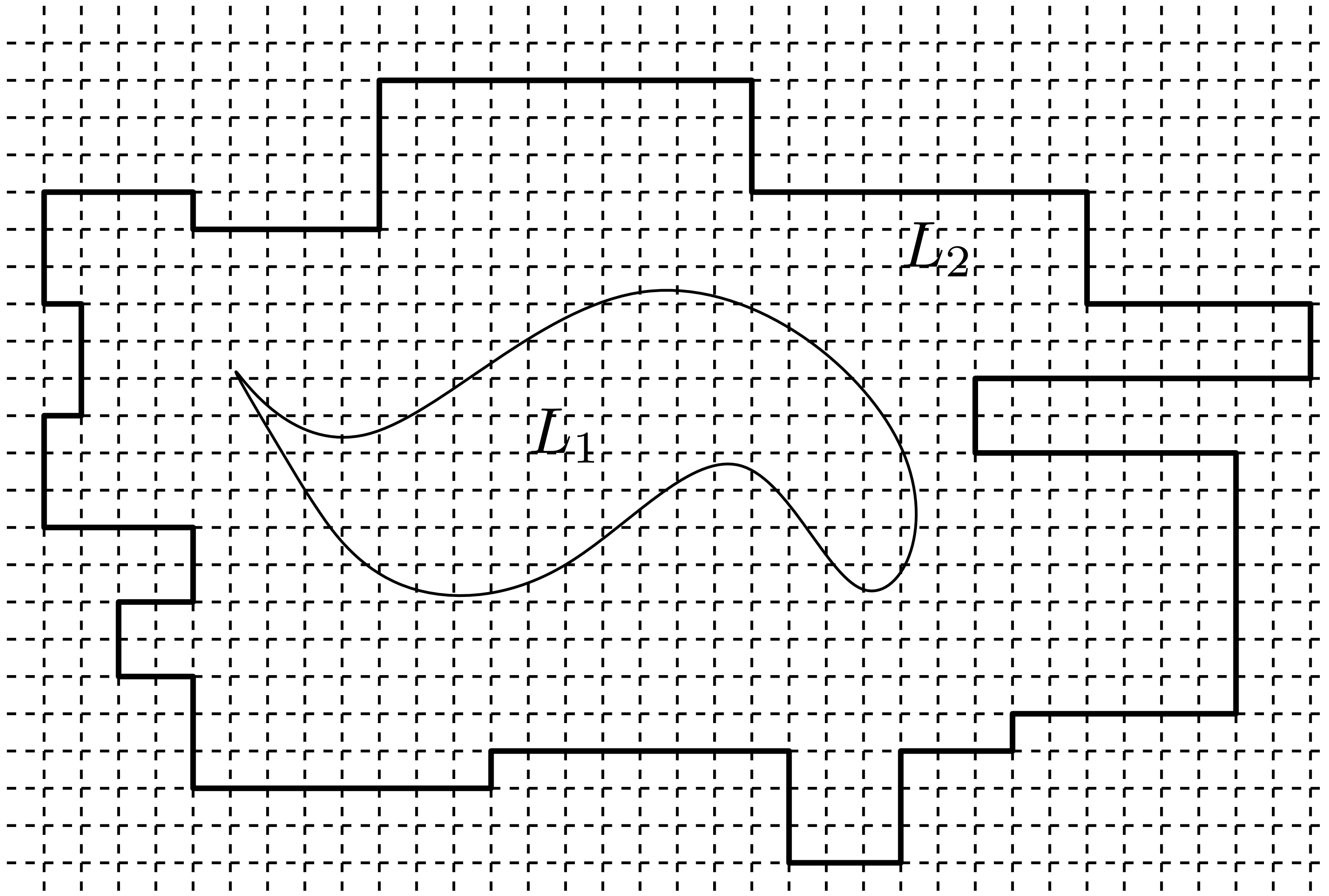
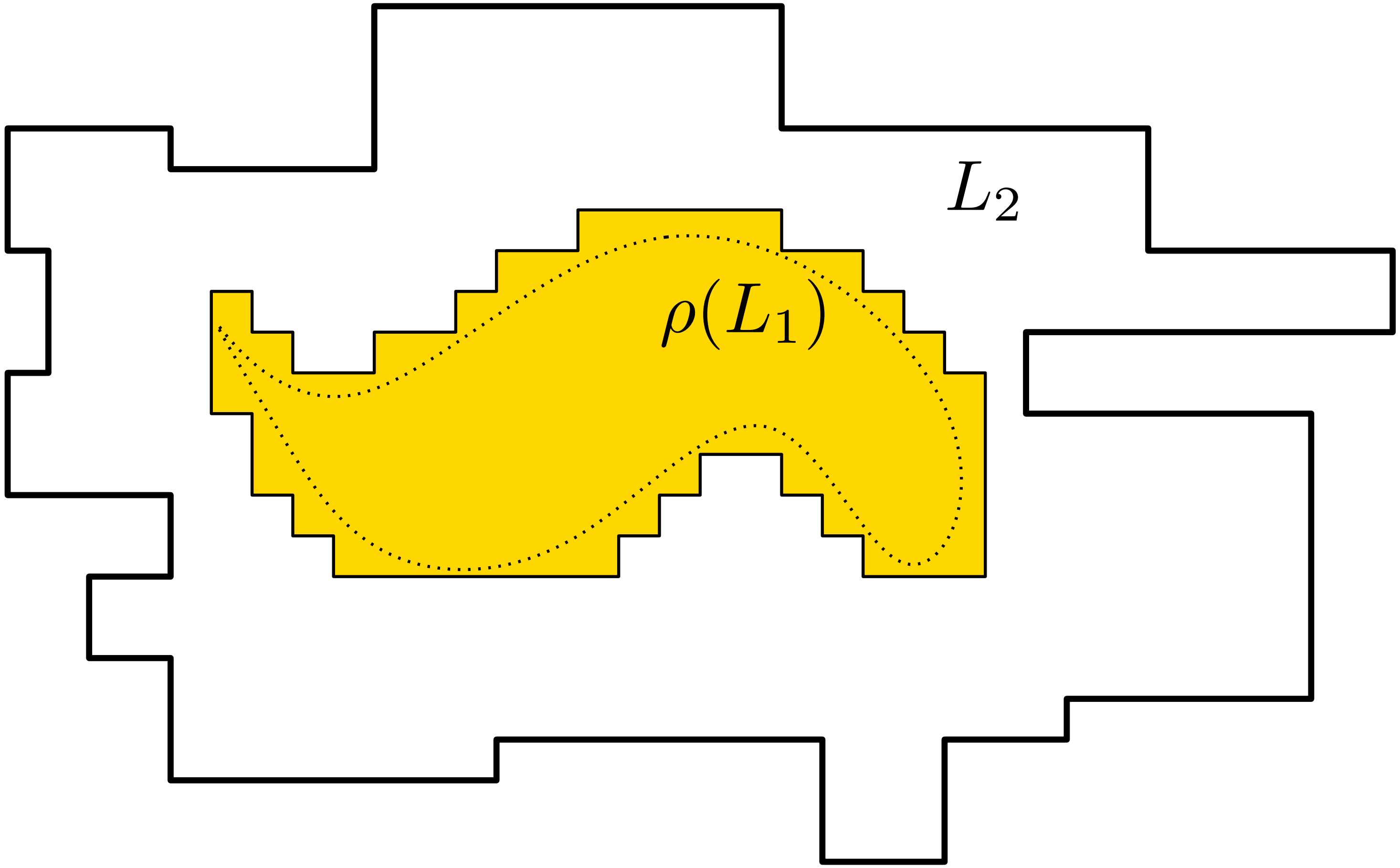
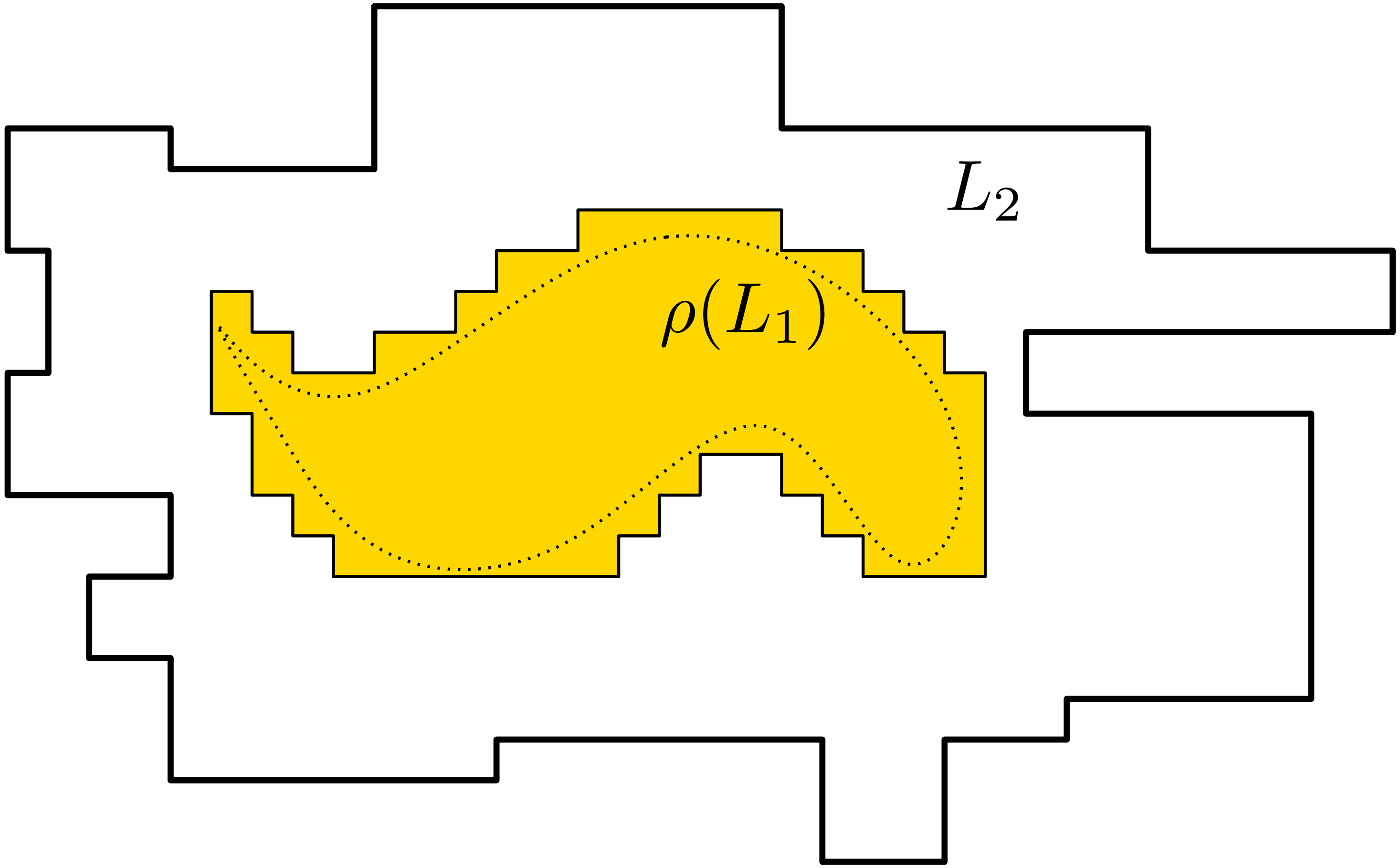
joint work with Pedro Valero (IMDEA) and Francesco Ranzato (Uni. Padova)


L_1

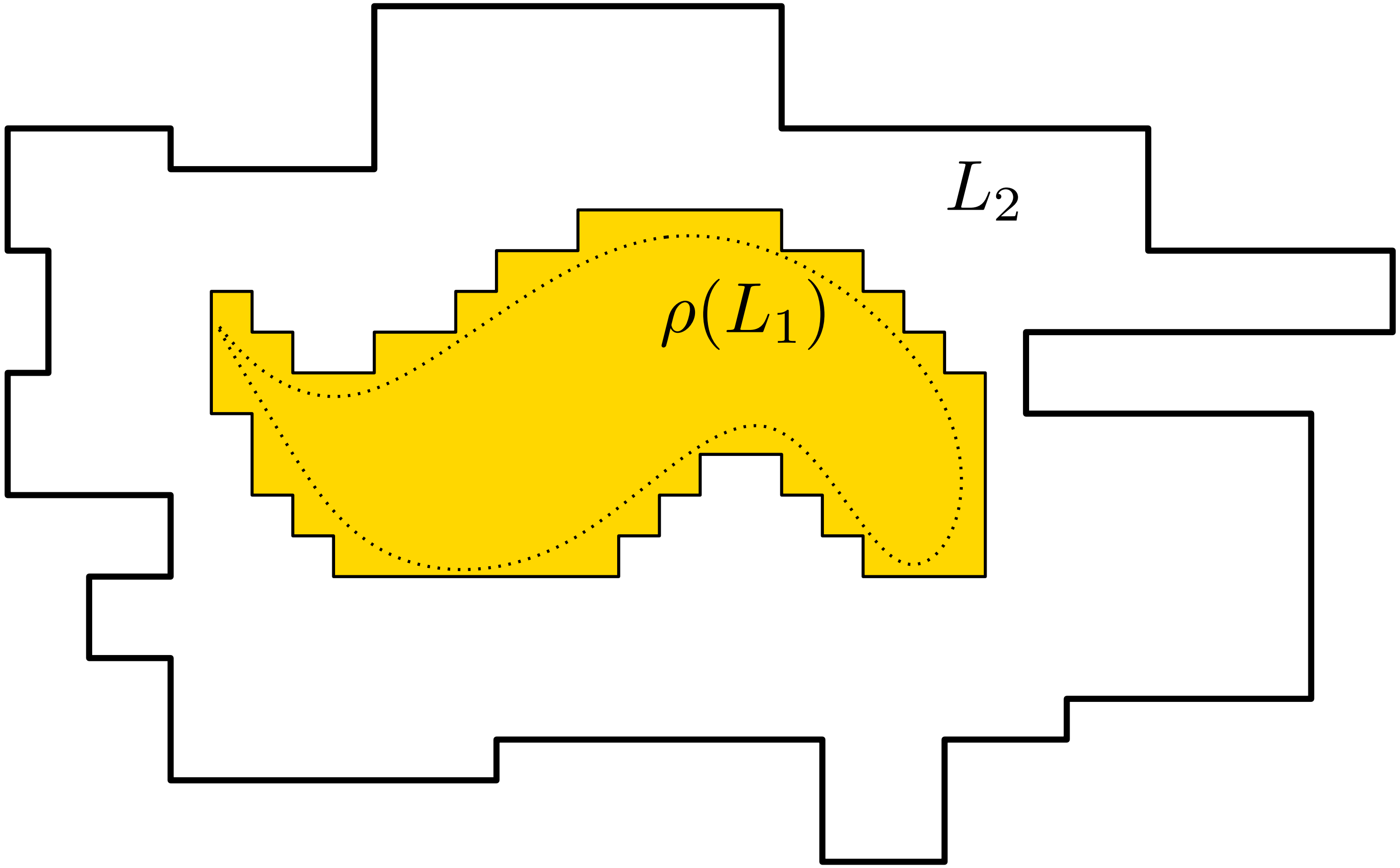

L_2
L_1


L_2
L_1


L_2
$\rho(L_1)$



- ρ is extensive: $L \subseteq \rho(L)$
- ρ is monotone: $L \subseteq L'$ implies $\rho(L) \subseteq \rho(L')$
- ρ is idempotent: $\rho(\rho(L)) = \rho(L)$

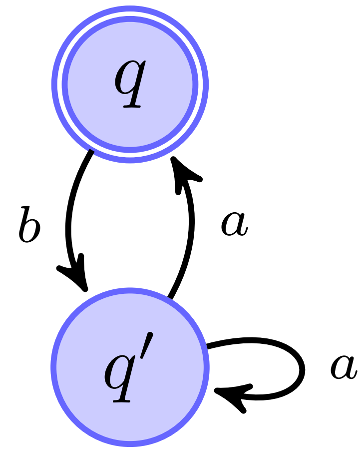


Assume $\rho(L_2) = L_2$, then

$$L_1 \subseteq L_2 \text{ iff } \rho(L_1) \subseteq L_2$$

$$\rho(L_2) = L_2$$

L_1 as a fixed point



$$\mathrm{lfp}\left(\lambda\begin{pmatrix} X_q \\ X_{q'} \end{pmatrix}.\begin{pmatrix} \{\epsilon\} & \cup & bX_{q'} \\ \varnothing & \cup & aX_q \cup aX_{q'} \end{pmatrix}\right)$$

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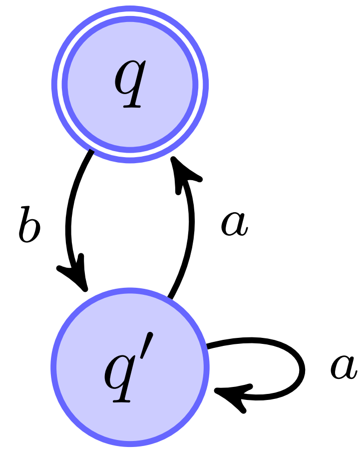

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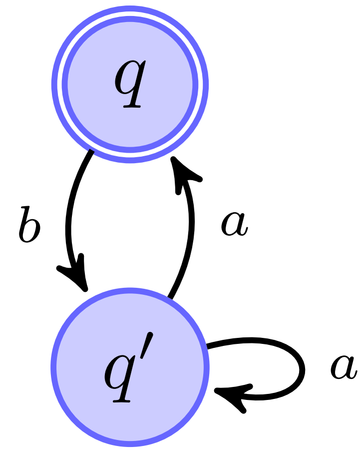

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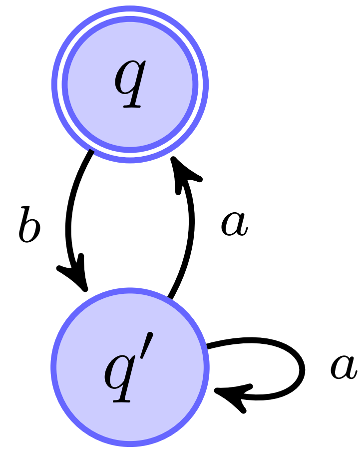

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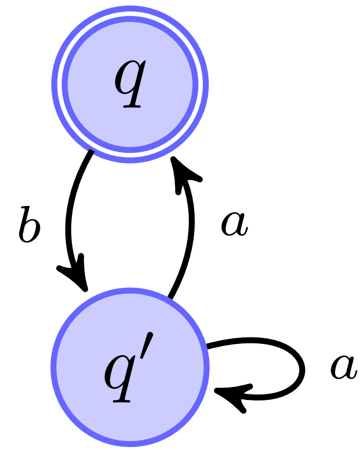

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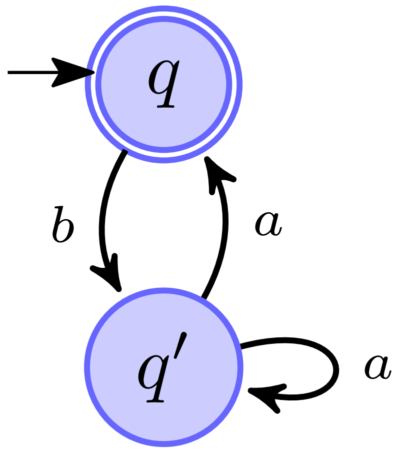

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$$L_1 = L_q = (ba^+)^* = \langle \mathrm{lfp}\, \lambda X.\, b \cup Fn(X) \rangle_q$$

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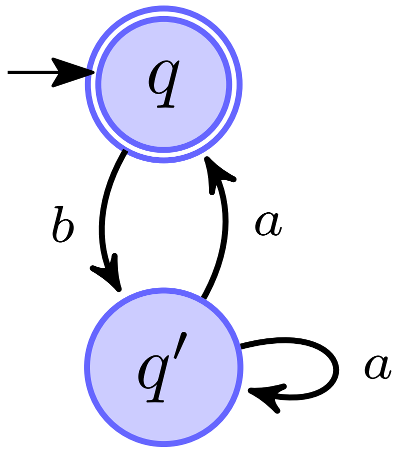

b Fn

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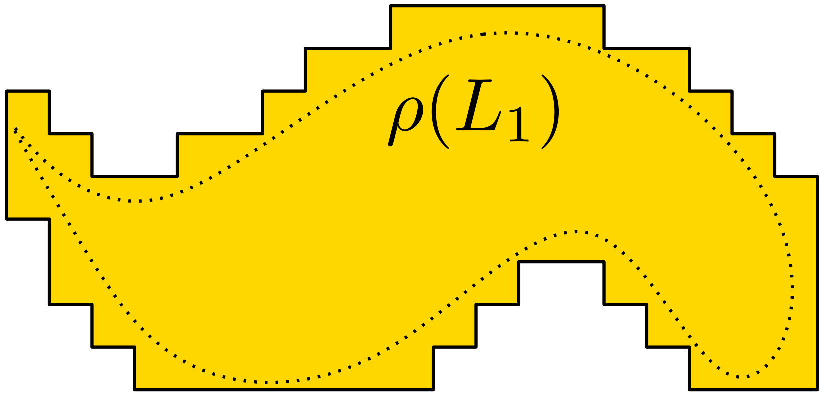

$$\rho(L_1) = \rho\big(\mathrm{lfp}\,\lambda X.\, b \cup Fn(X)\big)$$

if ρ is complete for Fn (i.e. $\rho\ Fn\ \rho = \rho\ Fn$) then

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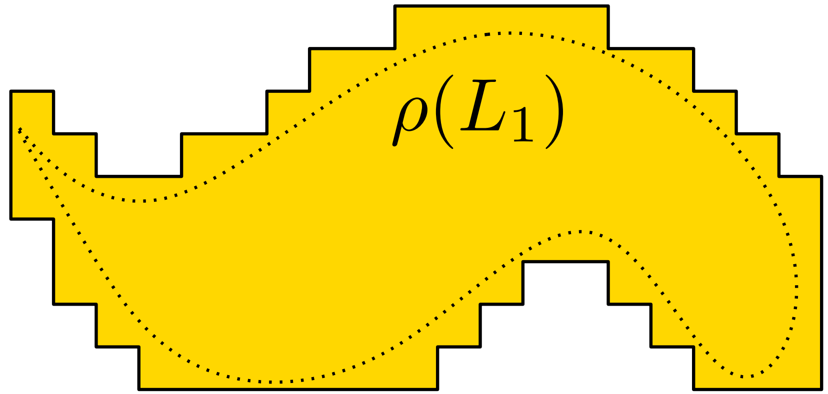

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ρ is complete for Fn

Wanted

$\rho(L_2) = L_2$

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Quasiorder induced ρ

Given a quasiorder $\leqslant$ on words, define

$$\rho_{\leqslant}(X) = \{y \mid \exists x \in X : x \leqslant y\}$$

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Example of quasiorder

Nerode left quasiorder relative to L_2 [Varricchio,deLuca'94]

$$x \leqslant^{l}_{L_2} y \Leftrightarrow L_2\, x^{-1} \subseteq L_2\, y^{-1}$$

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Nerode left quasiorder relative to $Lx^{-1} = \{w \mid wx \in L\}$

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$\leqslant^{l}_{L_2}$ is decidable if L_2 is regular

Deciding $L_1 \subseteq L_2$ using Nerode's left qo $\leqslant^{l}_{L_2}$

$L_1 \subseteq L_2$ iff $\rho_{\leqslant^{l}_{L_2}}(L_1) \subseteq L_2$

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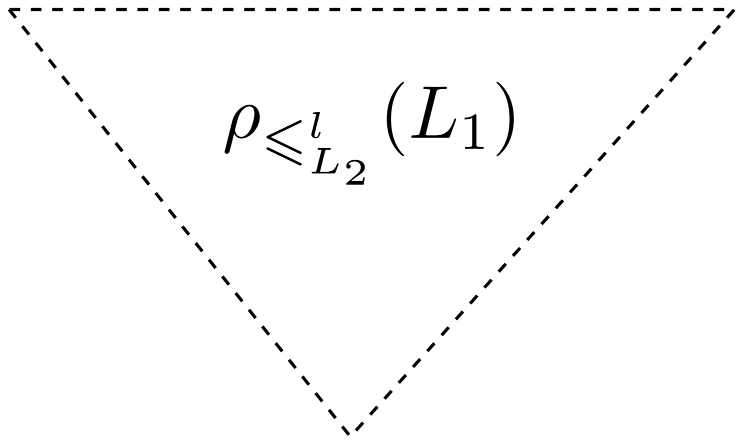

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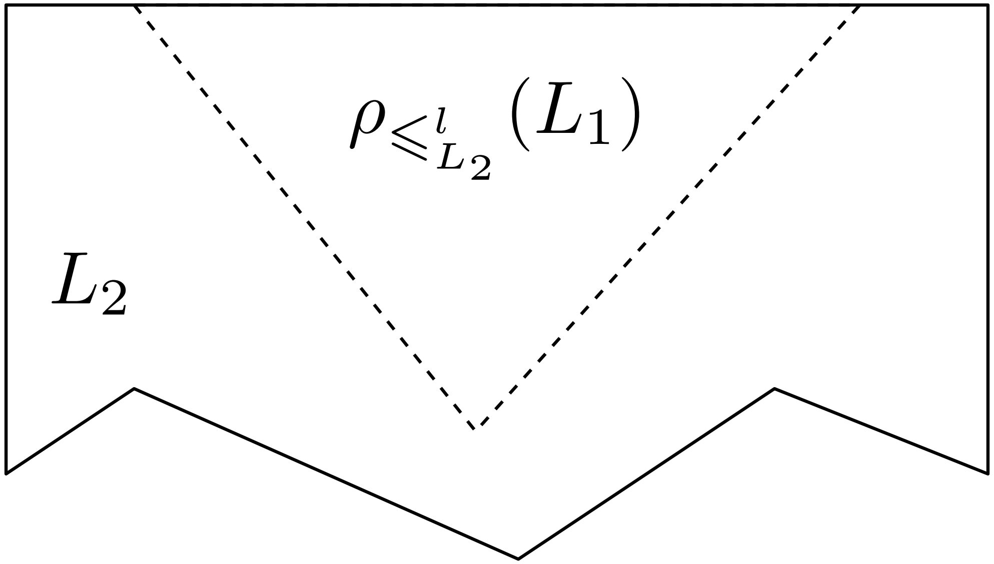

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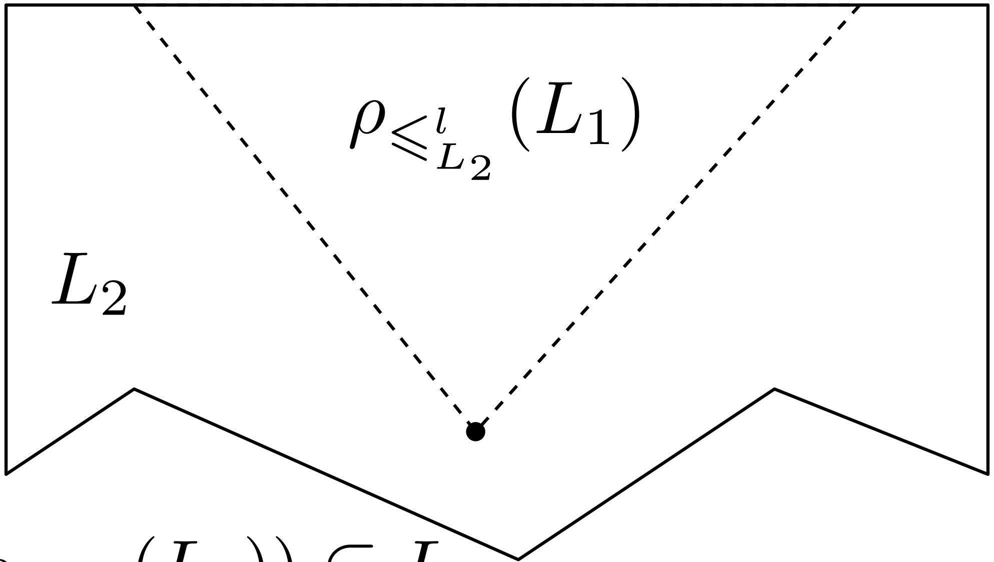

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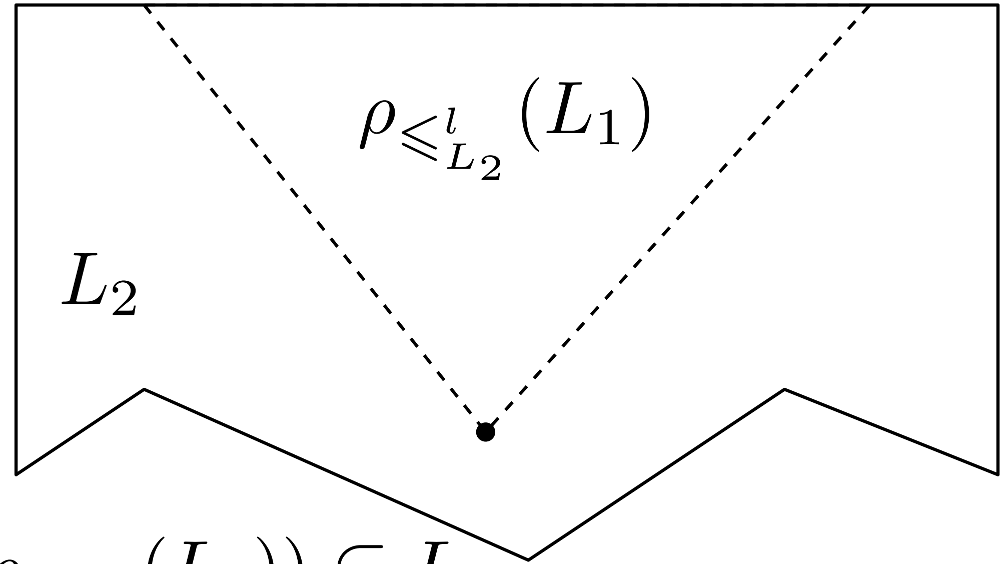

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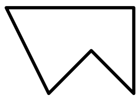

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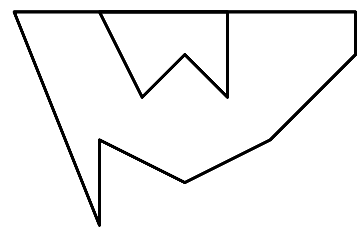

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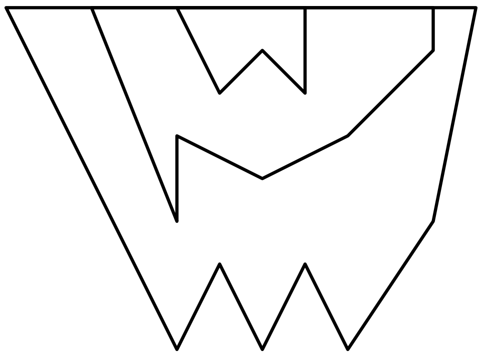

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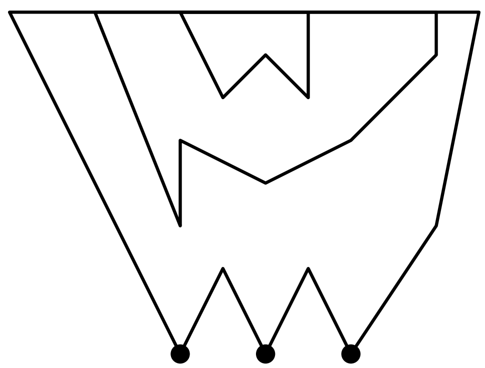

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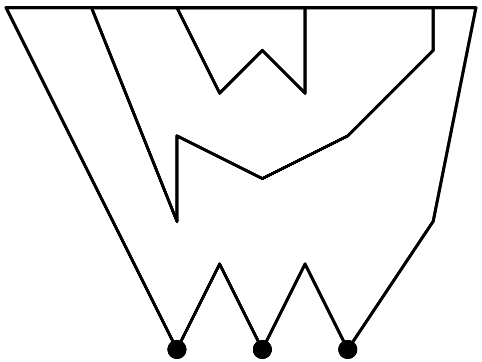

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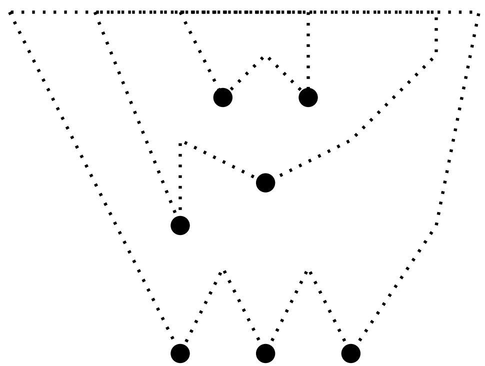

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Word-based antichain algorithm for $\mathcal{L}(\mathcal{A}) \subseteq L_2$

Data: FA $\mathcal{A} = \langle Q, \delta, q^0, F \rangle$

Data: L_2 regular

$\langle Y_q \rangle_{q \in Q} := \vec{\varnothing}$;

repeat

    $\langle X_q \rangle_{q \in Q} := \langle Y_q \rangle_{q \in Q}$;

    $\langle Y_q \rangle_{q \in Q} := \min_{\leqslant^l_{L_2}} \big(b \cup Fn(\langle Y_q \rangle_{q \in Q}) \big)$;

until $\rho_{\leqslant^l_{L_2}}(\langle Y_q \rangle_{q \in Q}) \subseteq \rho_{\leqslant^l_{L_2}}(\langle X_q \rangle_{q \in Q})$;

forall $u \in Y_{q^0}$ **do**

    if $u \notin L_2$ **then return** *false*;

return *true*;

What else?

Nerode left quasiorder relative to L_2

[Varricchio,deLuca'94]

$$x \leqslant^{l}_{L_2} y \Leftrightarrow L_2\, x^{-1} \subseteq L_2\, y^{-1}$$

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State based quasiorder

Let $\mathcal{A}_2 = \langle Q_2, \delta_2, q_2^0, F_2 \rangle$ be an automaton for L_2

$$x \leqslant^l_{\mathcal{A}_2} y \Leftrightarrow \mathrm{pre}_x(F_2) \subseteq \mathrm{pre}_y(F_2)$$

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1. $\leqslant^l_{\mathcal{A}_2} \cap (L_2 \times \overline{L_2}) = \varnothing$
2. $x \leqslant^l_{\mathcal{A}_2} y$ implies $a\, x \leqslant^l_{\mathcal{A}_2} a\, y$ for all a
3. $\leqslant^l_{\mathcal{A}_2}$ is a well-quasiorder
4. $\leqslant^l_{\mathcal{A}_2}$ is decidable

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> $\leqslant^{l}_{\mathcal{A}_2}$ is left L_2 consistent decidable wqo

Word-based antichain algorithm for $\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$

Data: FA $\mathcal{A}_1 = \langle Q_1, \delta_1, q_1^0, F_1 \rangle$
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until $\rho_{\leqslant^l_{\mathcal{A}_2}}(\langle Y_q \rangle_{q \in Q_1}) \subseteq \rho_{\leqslant^l_{\mathcal{A}_2}}(\langle X_q \rangle_{q \in Q_1})$;
forall $u \in Y_{q_1^0}$ **do**
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Word-based antichain algorithm for $\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$

Data: FA $\mathcal{A}_1 = \langle Q_1, \delta_1, q_1^0, F_1 \rangle$
Data: FA $\mathcal{A}_2 = \langle Q_2, \delta_2, q_2^0, F_2 \rangle$
1 $\langle Y_q \rangle_{q \in Q_1} := \vec{\varnothing}$;

State based quasiorder

Let $\mathcal{A}_2 = \langle Q_2, \delta_2, q_2^0, F_2 \rangle$ be an automaton for L_2

$$x \leqslant^l_{\mathcal{A}_2} y \Leftrightarrow \mathrm{pre}_x(F_2) \subseteq \mathrm{pre}_y(F_2)$$

7 **if** $u \notin L(\mathcal{A}_2)$ **then return** *false*;
8 **return** *true*;

Ditching words altogether

State-based antichain algorithm for $\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$

Data: FA $\mathcal{A}_1 = \langle Q_1, \delta_1, q_1^0, F_1 \rangle$
Data: FA $\mathcal{A}_2 = \langle Q_2, \delta_2, q_2^0, F_2 \rangle$

$\langle Y_q \rangle_{q \in Q_1} := \vec{\varnothing}$;
repeat
    $\langle X_q \rangle_{q \in Q_1} := \langle Y_q \rangle_{q \in Q_1}$;
    $\langle Y_q \rangle_{q \in Q_1} := [\ldots]$;
until $Y_q \subseteq^{\forall\exists} X_q$, *for all* $q \in Q_1$;
forall $s \in Y_{q_1^0}$ **do**
    if $q_2^0 \notin s$ **then return** *false*;
return *true*;

$$[\ldots] = \min_{\subseteq^{\forall\exists}} \{\mathrm{pre}_a^{\mathcal{A}_2}(s) \mid \exists a \in \Sigma,\ q' \in \delta_1(q,a), s \in X_{q'}\} \cup F_2$$

- Same as antichain algo
- Derived from general solution instantiated to $\leqslant^{l}_{\mathcal{A}_2}$
- Variants of the antichain algorithm as instantiations
- Simulation enhanced qo [2]
- Nerode's qo induce the coarsest qo based abstractions

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- Derived from general solution instantiated to $\leqslant^{l}_{\mathcal{A}_2}$
- Variants of the antichain algorithm as instantiations
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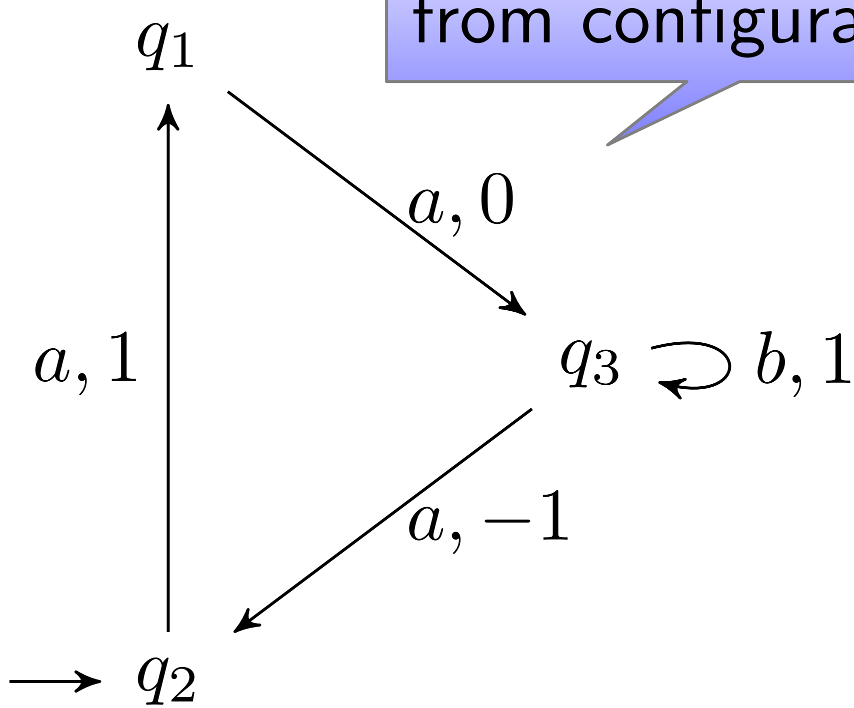
$\leqslant^{l}_{L_2}$ and $\leqslant^{r}_{L_2}$ are wqo's if L_2 is regular

L_2 is a one-counter net trace set

$L_2 = \mathcal{L}(\mathcal{O}_2)$ is the set of traces from configuration $q_2\,0$



The right Nerode qo for L_2 ($\leqslant^r_{L_2}$) is a wqo but it's undecidable

$u \leqslant^r_{\mathcal{O}_2} v$ iff $m_u \leq m_v$ where $m_u, m_v \in (\mathbb{N} \cup \{\bot\})^3$

$u \leqslant^r_{\mathcal{O}_2} v$ is a right L_2 consistent decidable wqo

Decision procedure for $\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{O}_2)$ using $\leqslant^r_{\mathcal{O}_2}$

Word-based antichain algorithm for $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{O}_2)$

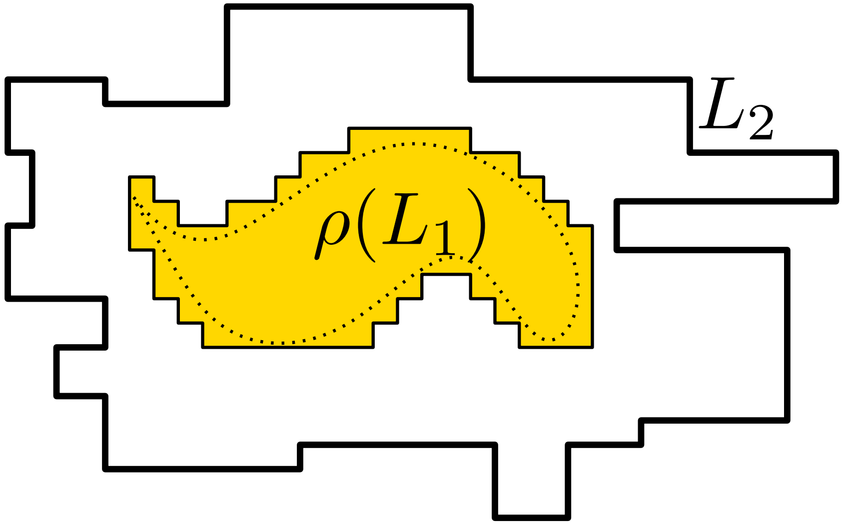
Data: FA $\mathcal{A} = \langle Q, \delta, q^0, F \rangle$
Data: OCN $\mathcal{O}_2$

```
⟨Y_q⟩_{q∈Q} := ∅⃗;
repeat
    ⟨X_q⟩_{q∈Q} := ⟨Y_q⟩_{q∈Q};
    ⟨Y_q⟩_{q∈Q} := min_{⩽^r_{O_2}}(b ∪ Gn(⟨Y_q⟩_{q∈Q}));
until ρ_{⩽^r_{O_2}}(⟨Y_q⟩_{q∈Q}) ⊆ ρ_{⩽^r_{O_2}}(⟨X_q⟩_{q∈Q});
forall u ∈ Y_{q^0} do
    if u ∉ L_2 then return false;
return true;
```

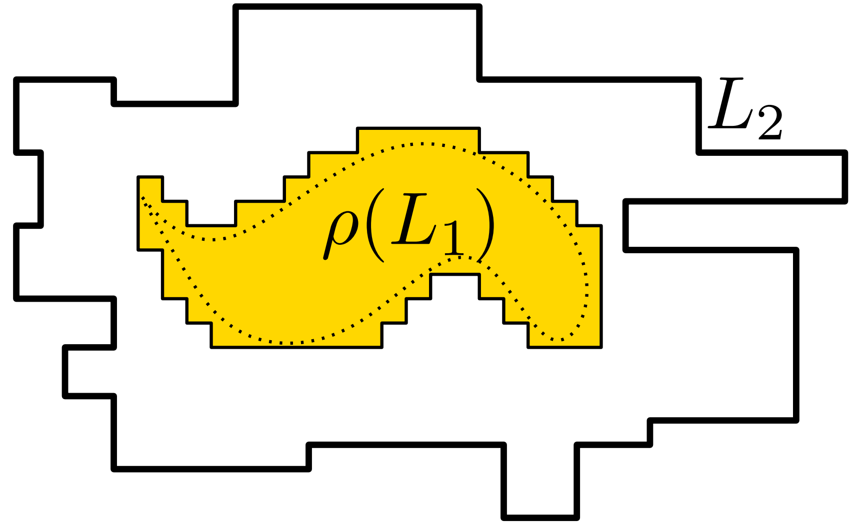
So far. . .

$$\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$$

$$\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{O}_2)$$



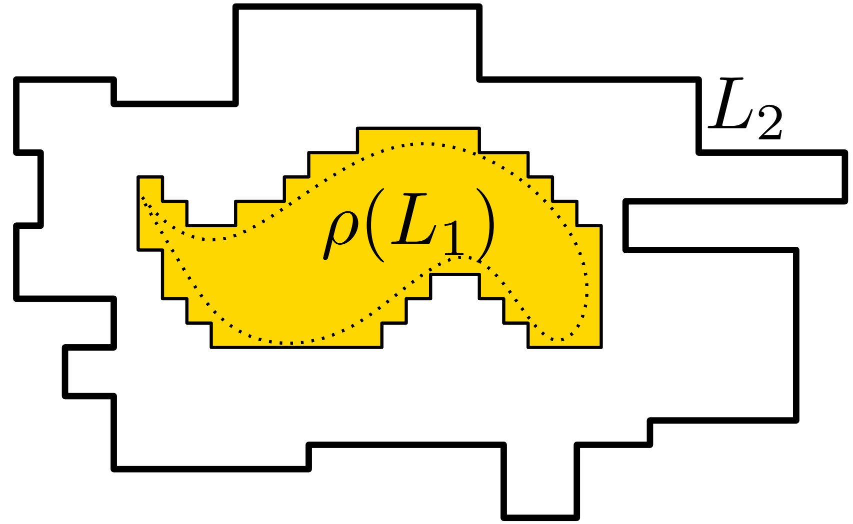
$$
\begin{aligned}
\rho(L_1) &= \rho\big(\mathrm{lfp}\,\lambda X.\, b \cup Fn(X)\big) \\
&= \mathrm{lfp}\,\lambda X.\, \rho\big(b \cup Fn(X)\big)
\end{aligned}
$$



$\rho \; Fn \; \rho = \rho \; Fn$

$\rho(L_1)$ … $Fn(X)\big)$

$= \mathrm{lfp}\, \lambda X.\, \rho\big(b \cup Fn(X)\big)$



$$\rho(L_1) \qquad \rho\ Fn\ \rho = \rho\ Fn \qquad Fn(X)\big)$$

$$= \operatorname{lfp} \lambda X.\, \rho\big(b \cup Fn(X)\big)$$

$$\operatorname{lfp}\left(\lambda \begin{pmatrix} X_q \\ X_{q'} \end{pmatrix} . \begin{pmatrix} \{\epsilon\} & \cup & bX_{q'} \\ \varnothing & \cup & aX_q \cup aX_{q'} \end{pmatrix}\right)$$

When L_1 is regular, ρ is complete for $\lambda X.\, aX$ for all a

Quasiorders for the case $\mathcal{L}(\mathcal{G}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$

$Fn_{\mathcal{G}_1}$ is more general: from aX, Xa, aXb, XX

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1. $\leqslant \cap\, (L_2 \times \overline{L_2}) = \varnothing$
2. $x \leqslant y$ implies $a\,x \leqslant a\,y$ **and** $x\,a \leqslant y\,a$ for all a

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Myhill quasiorder relative to L_2 [Varricchio, deLuca'94]

$$x \leqslant_{L_2} y \Leftrightarrow \{(u,v) \mid u\,x\,v \in L_2\} \subseteq \{(u,v) \mid u\,y\,v \in L_2\}$$

Quasiorders for the case $\mathcal{L}(\mathcal{G}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$

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State based quasi order [Netys'15]

$$x \leqslant_{\mathcal{A}_2} y \Leftrightarrow \{(q,q') \mid q \overset{x}{\rightsquigarrow} q'\} \subseteq \{(q,q') \mid q \overset{y}{\rightsquigarrow} q'\}$$

Greatest fixpoint based approach

$$\mathrm{lfp}\,\lambda X.\, b \cup Fn(X) \subseteq L_2 \text{ iff } b \subseteq \mathrm{gfp}\,\lambda X.\, L_2 \cap \widetilde{Fn}(X)$$

$\cup$ becomes $\cap$

$\lambda X.\, aX$ becomes $\lambda X.\, a^{-1}X$

$\rho\ Fn\ \rho = \rho\ Fn$ becomes $\rho\ \widetilde{Fn}\ \rho = \widetilde{Fn}\ \rho$

$\mathrm{gfp}\,\lambda X.\, \rho(L_2 \cap \widetilde{Fn}(X))$ stepwise equal $\mathrm{gfp}\,\lambda X.\, L_2 \cap \widetilde{Fn}(X)$

Greatest fixpoint based approach

Does not terminate in general

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Greatest fixpoint based approach

Does not terminate in general

Does always terminate [1]

$$\mathrm{lfp}\,\lambda X.\, b \cup Fn(X) \subseteq L_2 \text{ iff } b \subseteq \mathrm{gfp}\,\lambda X.\, L_2 \cap \widetilde{Fn}(X)$$

$\cup$ becomes $\cap$

$\lambda X.\, aX$ becomes $\lambda X.\, a^{-1}X$

$\rho\; Fn\; \rho = \rho\; Fn$ becomes $\rho\; \widetilde{Fn}\; \rho = \widetilde{Fn}\; \rho$

$\mathrm{gfp}\,\lambda X.\, \rho(L_2 \cap \widetilde{Fn}(X))$ stepwise equal $\mathrm{gfp}\,\lambda X.\, L_2 \cap \widetilde{Fn}(X)$

Conclusion

Promising framework for language inclusion

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Extension to trees, ω-languages, timed languages,. . .

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Promising framework for language inclusion

Extension to trees, ω-languages, timed languages,. . .

3.19 Notes on regularity and well quasi-ordering

Mizuhito Ogawa (JAIST – Ishikawa, JP)

In [2], Ehrenfeucht et al. showed that a set L of finite words (over finite alphabet) is regular if and only if L is $\leq$-closed under some monotone well quasi-order $\leq$ over finite words. This note briefly surveys extensions to finite trees and ω-words [4]. They are obtained by similar proofs by modifying the standard congruence in Myhill-Nerode theorem to those in [3, 1]. The extensions are,

1. a tree language L is regular if and only if L is $\leq$-closed under some monotone well quasi-order $\leq$ over finite trees.
2. an ω-language L is regular if and only if L is $\preceq$-closed under a *periodic* extension $\preceq$ of some monotone WQO over finite words, and
3. an ω-language L is regular if and only if L is $\preceq$-closed under a WQO $\preceq$ over ω-words that is a *continuous* extension of some monotone WQO over finite words.

On the Language Inclusion Problem for Timed Automata: Closing a Decidability Gap

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