Permutation-based Combinatorial Optimization Problems under the Microscope

Josu Ceberio

Intelligent Systems Group
Department of Computer Science and Artificial Intelligence
University of the Basque Country (UPV/EHU)

Intelligent Systems Group

eman ta zabal zazu
Universidad del País Vasco Euskal Herriko Unibertsitatea

Permutation-based Problems

Combinatorial Optimization Problems

Whose solutions are represented as **permutations**

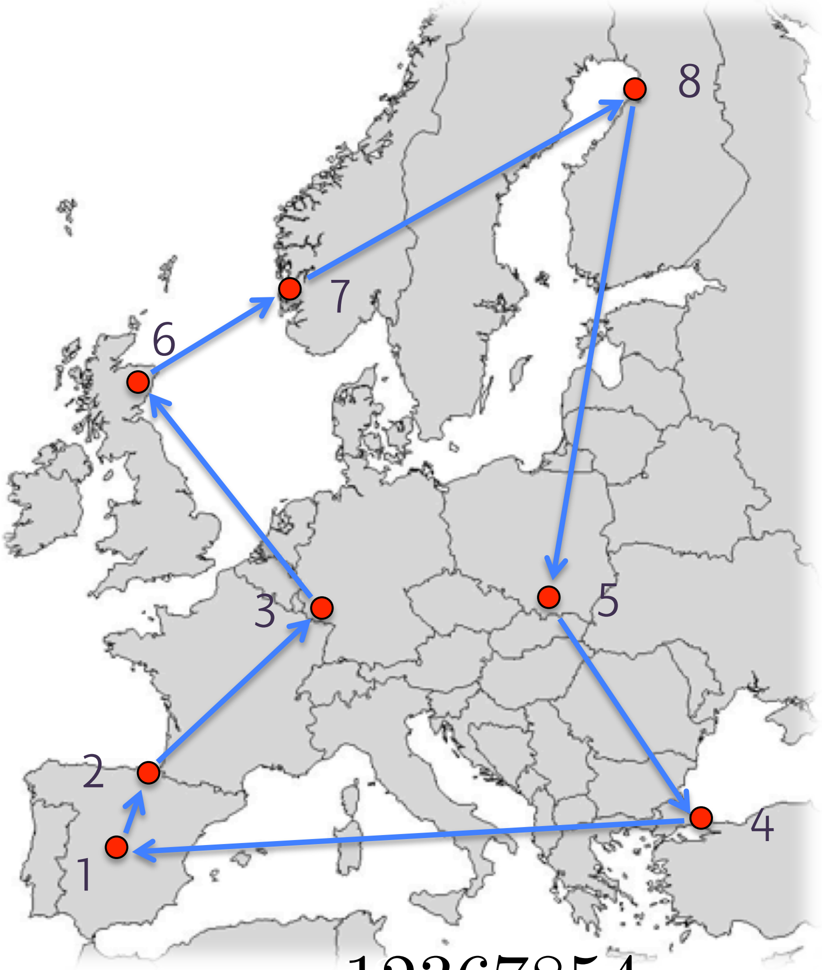
The search space consist of $n!$ solutions

$$8! = 40320$$

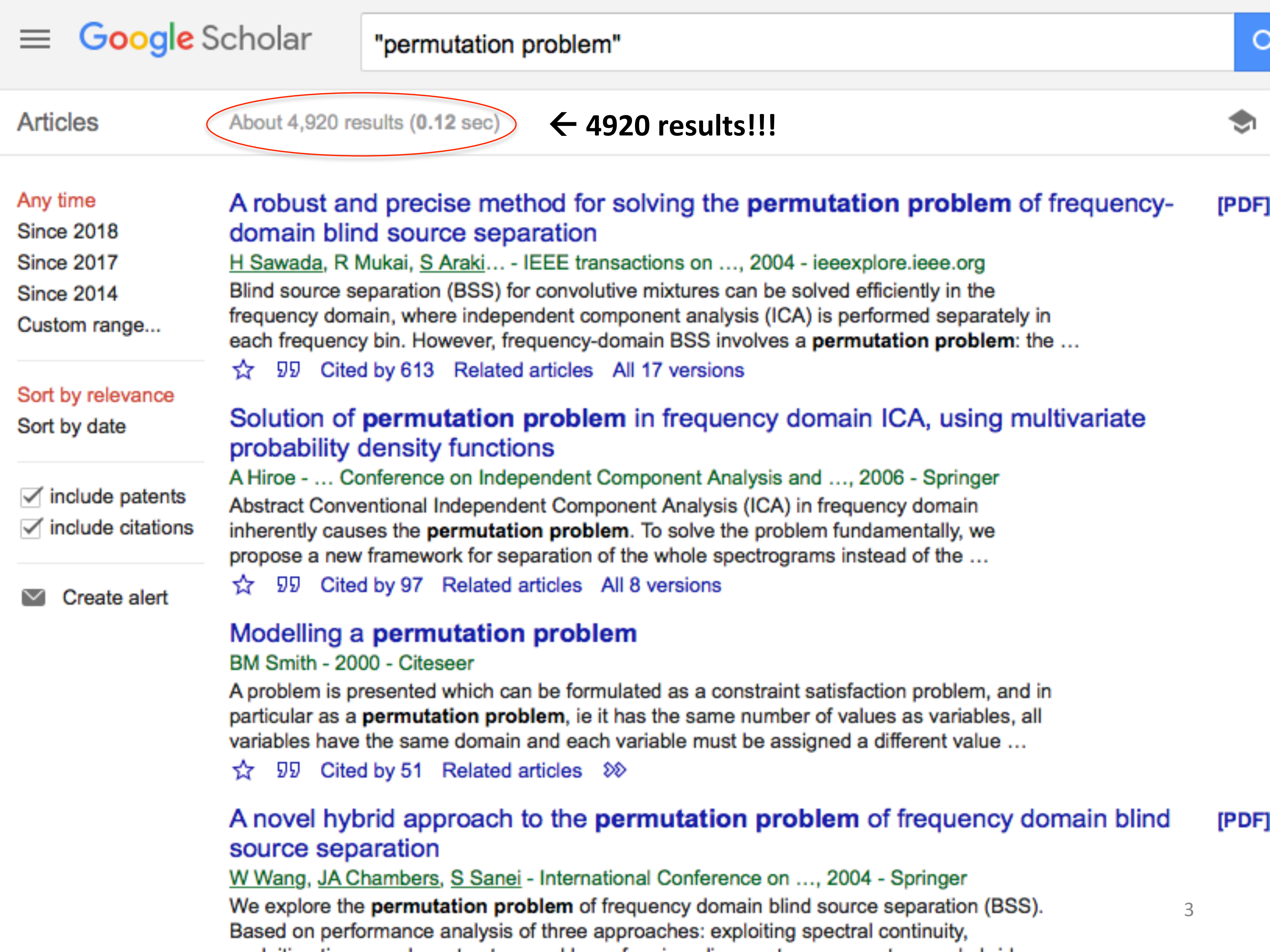
$$20! = 2.43 \times 10^{18}$$

NP-Hard in most of the cases

Travelling Salesman Problem (TSP)



$$\sigma = 12367854$$



About 4,920 results (0.12 sec)

← 4920 results!!!

A robust and precise method for solving the permutation problem of frequency-domain blind source separation [PDF]

H Sawada, R Mukai, S Araki… - IEEE transactions on …, 2004 - ieeexplore.ieee.org

Blind source separation (BSS) for convolutive mixtures can be solved efficiently in the frequency domain, where independent component analysis (ICA) is performed separately in each frequency bin. However, frequency-domain BSS involves a **permutation problem**: the …

☆ 99 Cited by 613 Related articles All 17 versions

Solution of permutation problem in frequency domain ICA, using multivariate probability density functions

A Hiroe - … Conference on Independent Component Analysis and …, 2006 - Springer

Abstract Conventional Independent Component Analysis (ICA) in frequency domain inherently causes the **permutation problem**. To solve the problem fundamentally, we propose a new framework for separation of the whole spectrograms instead of the …

☆ 99 Cited by 97 Related articles All 8 versions

Modelling a permutation problem

BM Smith - 2000 - Citeseer

A problem is presented which can be formulated as a constraint satisfaction problem, and in particular as a **permutation problem**, ie it has the same number of values as variables, all variables have the same domain and each variable must be assigned a different value …

☆ 99 Cited by 51 Related articles

A novel hybrid approach to the permutation problem of frequency domain blind source separation [PDF]

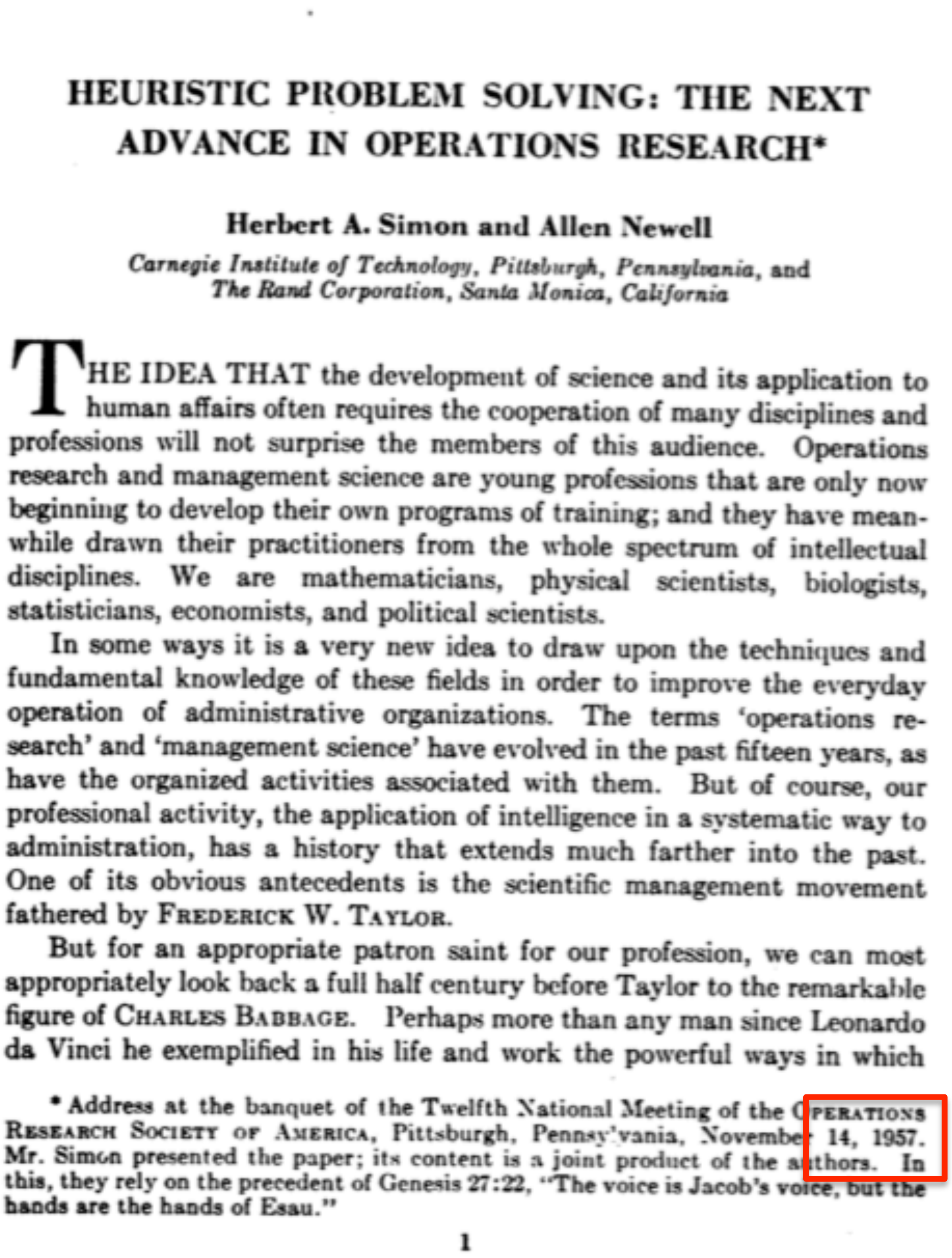
W Wang, JA Chambers, S Sanei - International Conference on …, 2004 - Springer

We explore the **permutation problem** of frequency domain blind source separation (BSS). Based on performance analysis of three approaches: exploiting spectral continuity,

Revised approaches…

- Branch & Bound
- Branch & Cut
- Linear Programming
- Genetic Algorithms
- Variable Neighborhood Search
- Variable Neighborhood Descent
- Memetic Algorithm
- Estimation of Distribution Algorithms
- Constructive Algorithms
- Local Search

- Ant Colony Optimization
- Tabu Search
- Scatter Search
- Genetic Programming
- Cutting Plane Algorithms
- Particle Swarm Optimization
- Simulated Annealing
- Cuckoo Search
- Differential Evolution
- Artificial Bee Colony Algorithm



HEURISTIC PROBLEM SOLVING: THE NEXT ADVANCE IN OPERATIONS RESEARCH*

Herbert A. Simon and Allen Newell

Carnegie Institute of Technology, Pittsburgh, Pennsylvania, and
The Rand Corporation, Santa Monica, California

THE IDEA THAT the development of science and its application to human affairs often requires the cooperation of many disciplines and professions will not surprise the members of this audience. Operations research and management science are young professions that are only now beginning to develop their own programs of training; and they have meanwhile drawn their practitioners from the whole spectrum of intellectual disciplines. We are mathematicians, physical scientists, biologists, statisticians, economists, and political scientists.

In some ways it is a very new idea to draw upon the techniques and fundamental knowledge of these fields in order to improve the everyday operation of administrative organizations. The terms 'operations research' and 'management science' have evolved in the past fifteen years, as have the organized activities associated with them. But of course, our professional activity, the application of intelligence in a systematic way to administration, has a history that extends much farther into the past. One of its obvious antecedents is the scientific management movement fathered by FREDERICK W. TAYLOR.

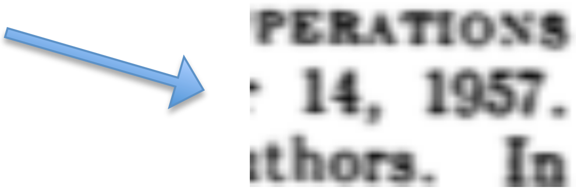
But for an appropriate patron saint for our profession, we can most appropriately look back a full half century before Taylor to the remarkable figure of CHARLES BABBAGE. Perhaps more than any man since Leonardo da Vinci he exemplified in his life and work the powerful ways in which

---

* Address at the banquet of the Twelfth National Meeting of the OPERATIONS RESEARCH SOCIETY OF AMERICA, Pittsburgh, Pennsylvania, November 14, 1957. Mr. Simon presented the paper; its content is a joint product of the authors. In this, they rely on the precedent of Genesis 27:22, "The voice is Jacob's voice, but the hands are the hands of Esau."

"…propose that a theory of heuristic (as opposed to algorithmic or exact) problem-solving should focus on ***intuition, insight and learning****."*

"In order to design algorithms practitioners should gain ***a deep insight into the structure of the problem*** *that is to be solved."* (Sorensen 2012).



Permutation Flowshop Scheduling Problem and Estimation of Distribution Algorithms

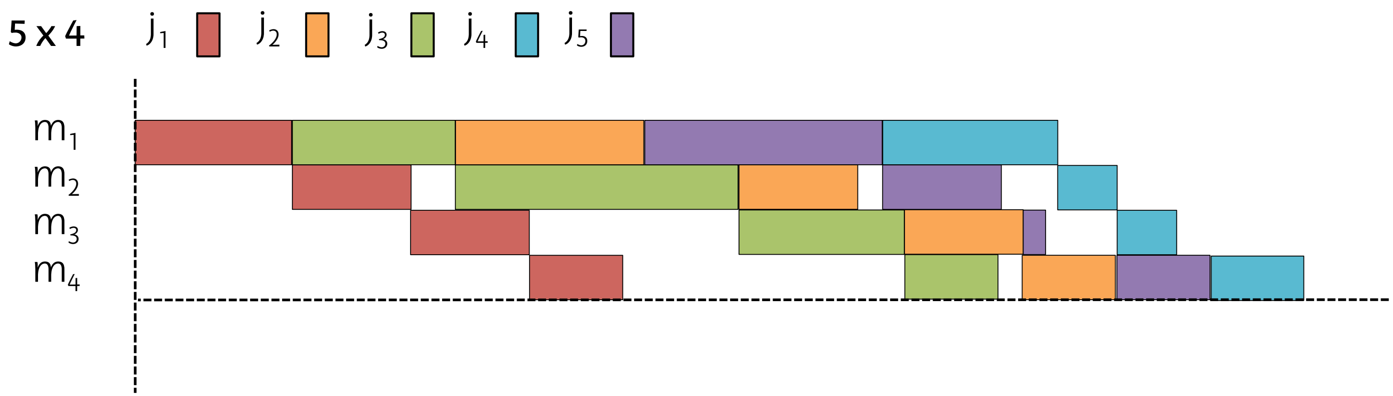
Example 1

Permutation Flowshop Scheduling Problem

- n jobs
- m machines
- processing times p_{ij}

Total flow time (TFT)

$$f(\sigma) = \sum_{i=1}^{n} c_{\sigma(i),m}$$



$$\sigma = 13254$$

Revised approaches…

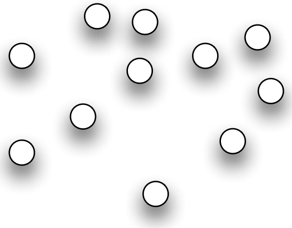
- Branch & Bound
- Branch & Cut
- Linear Programming
- Genetic Algorithms
- Variable Neighborhood Search
- Variable Neighborhood Descent
- Memetic Algorithm
- Estimation of Distribution Algorithms
- Constructive Algorithms
- Local Search

Why?

- Ant Colony Optimization
- Tabu Search
- Scatter Search
- Genetic Programming
- Cutting Plane Algorithms
- Particle Swarm Optimization
- Simulated Annealing
- Cuckoo Search
- Differential Evolution
- Artificial Bee Colony Algorithm

Estimation of distribution algorithms

Generate a set
of solutions

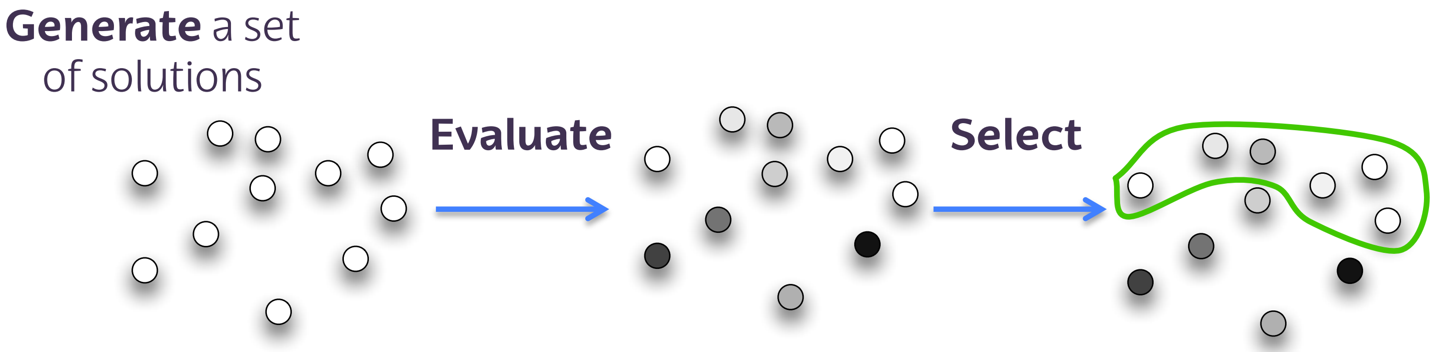


Estimation of distribution algorithms

Generate a set of solutions

Evaluate

Estimation of distribution algorithms

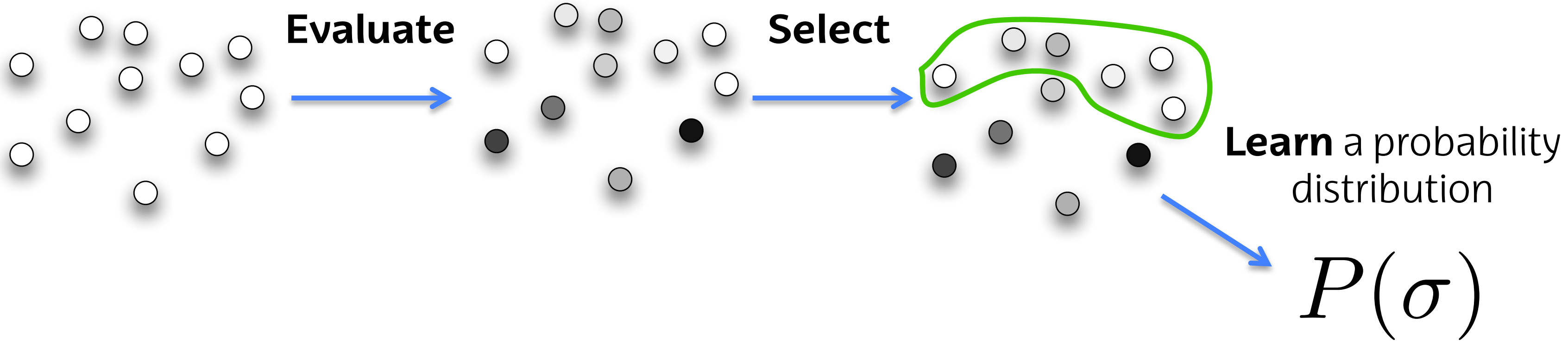


Generate a set of solutions

Evaluate

Select

Estimation of distribution algorithms



Generate a set of solutions

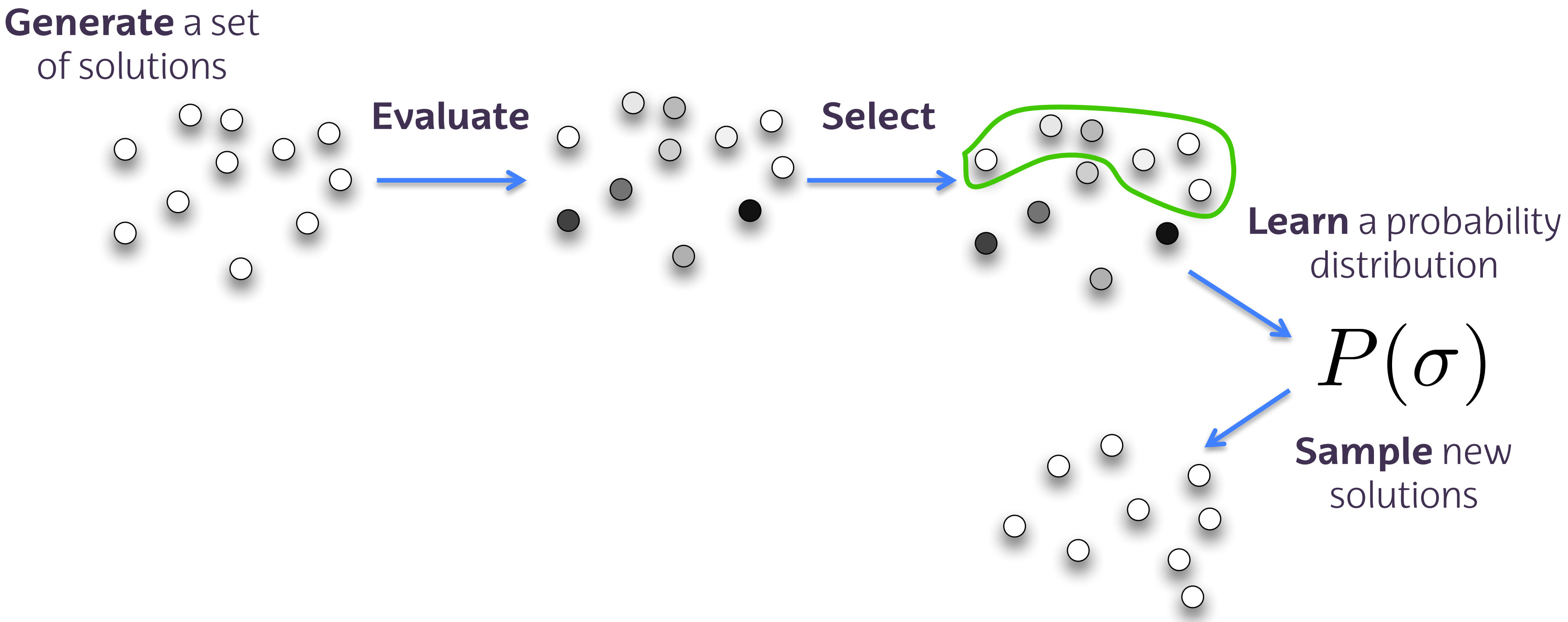
Evaluate

Select

Learn a probability distribution

$$P(\sigma)$$

Estimation of distribution algorithms



Generate a set of solutions

Evaluate

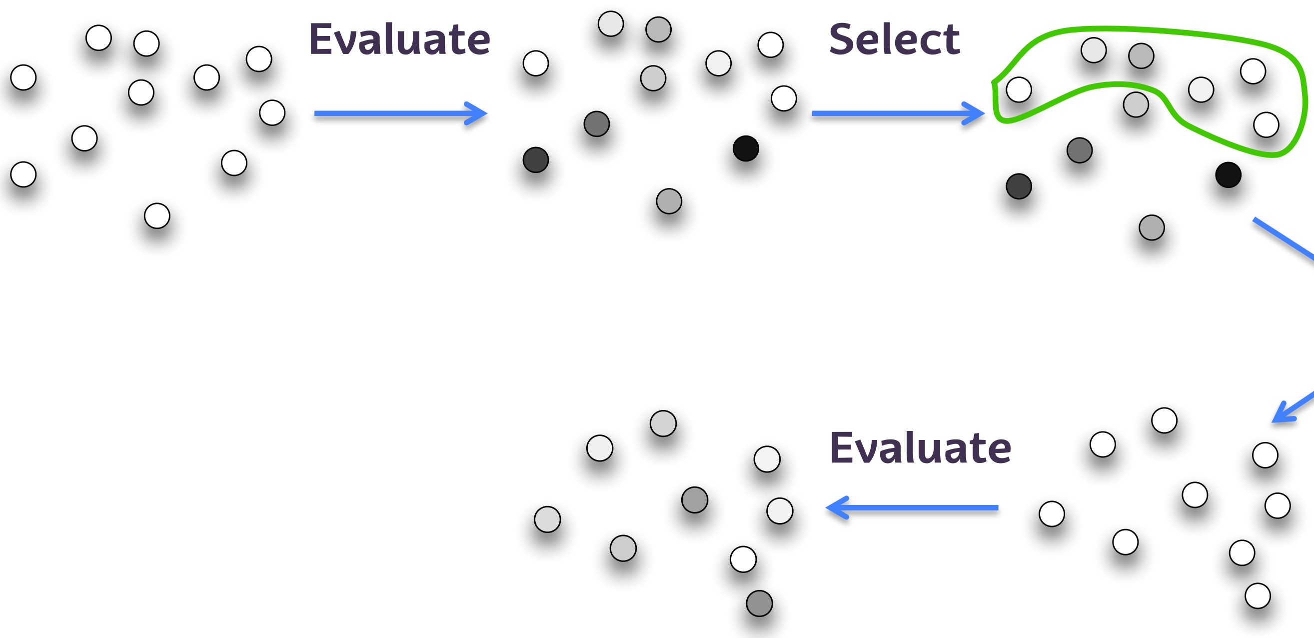
Select

Learn a probability distribution

$$P(\sigma)$$

Sample new solutions

Estimation of distribution algorithms



Generate a set of solutions

Evaluate

Select

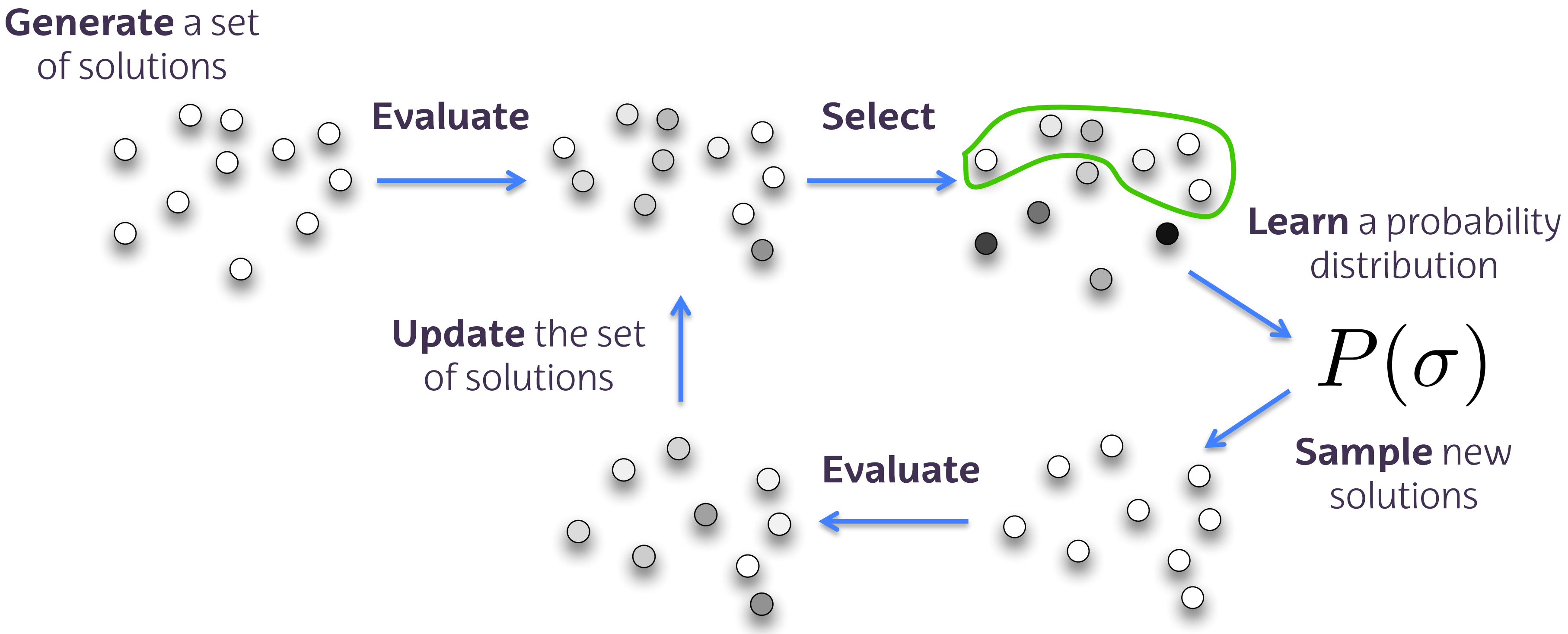
Learn a probability distribution

$$P(\sigma)$$

Sample new solutions

Evaluate

Estimation of distribution algorithms



Generate a set of solutions

Evaluate

Select

Learn a probability distribution

$$P(\sigma)$$

Sample new solutions

Evaluate

Update the set of solutions

EDAs reported in the literature

Combinatorial Problems

UMDA [Mühlenbein, 1998]
MIMIC [DeBonet, 1997]
FDA [Mühlenbein, 1999]
EBNA [Etxeberria, 1999]
BOA [Pelikan, 2000]
EHBSA [Tsutsui, 2003]
NHBSA [Tsutsui, 2006]
TREE [Pelikan, 2007]
REDA [Romero, 2009]

Ω

Continuous Problems

UMDA_c [Larrañaga, 2000]
MIMIC_c [Larrañaga, 2000]
EGNA [Larrañaga, 2000]
EMNA [Larrañaga, 2001]
IDEA [Bosman, 2000]

$\mathbb{R}^n$

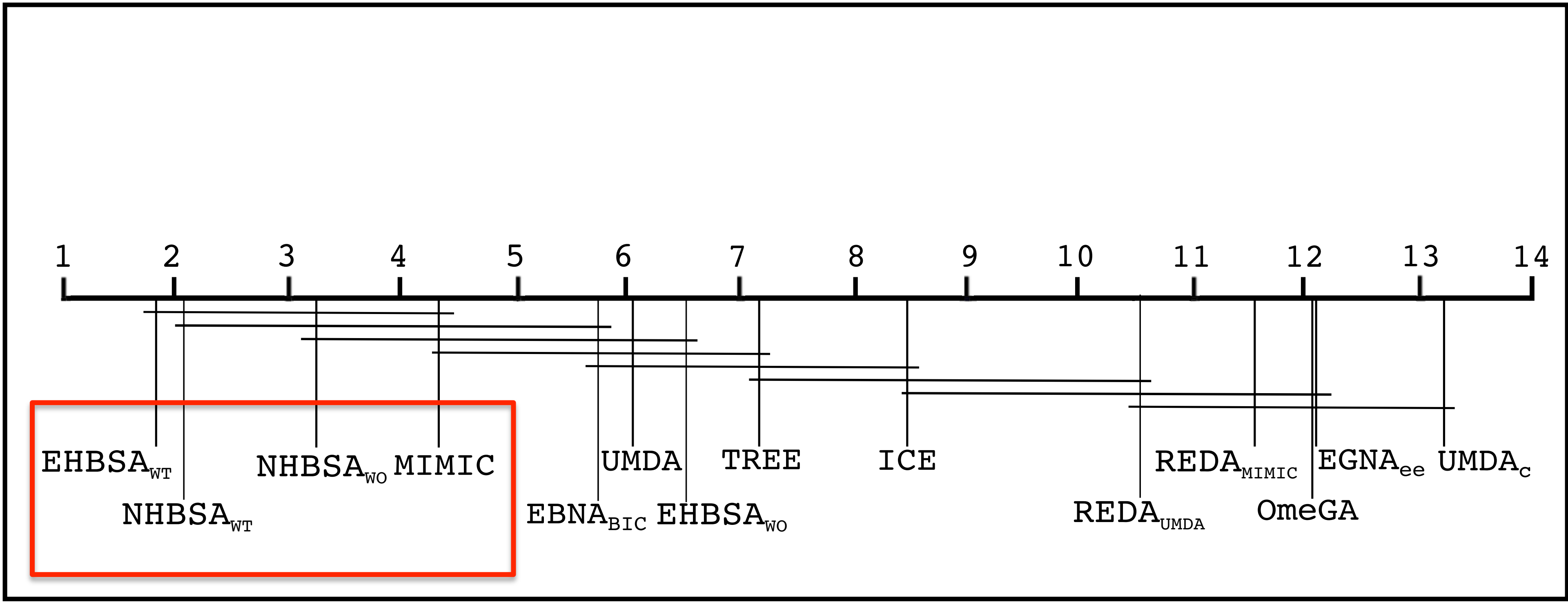
$\mathbb{S}_n$

Permutation Problems

IDEA-ICE [Bosman, 2001]

...

Experiments on
Permutation Flowshop Scheduling Problem



Univariate and bivariate models!!!

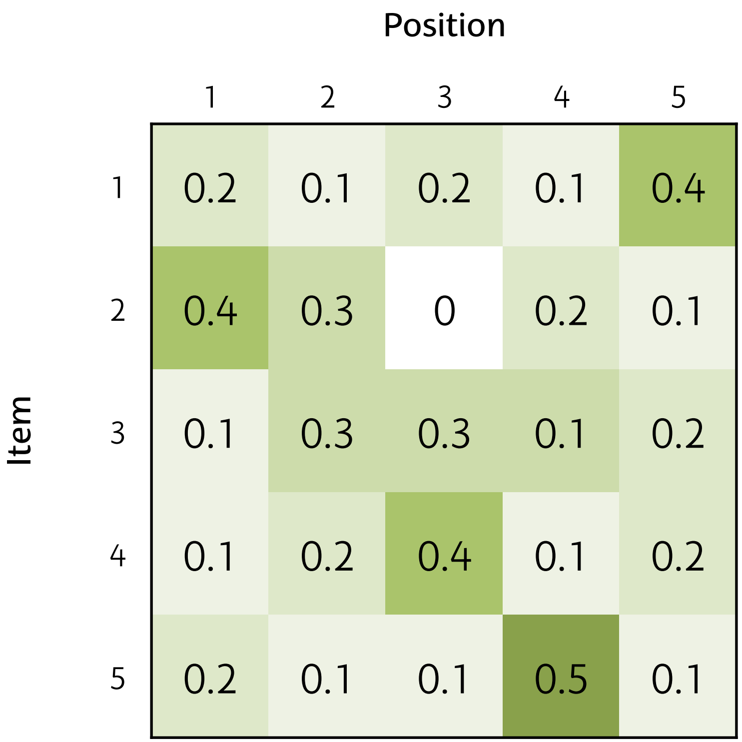
Experiments on
Permutation Flowshop Scheduling Problem

- Node and Edge Histogram-based Sampling Algorithms (EHBSA & NHBSA)
  (Tsutsui et al. 2002, Tsutsui et al. 2006)

Population

54123
42351
12354
24351
31452
23415
23451
25431
12543
53124

Node Histogram



Position

Item	1	2	3	4	5
1	0.2	0.1	0.2	0.1	0.4
2	0.4	0.3	0	0.2	0.1
3	0.1	0.3	0.3	0.1	0.2
4	0.1	0.2	0.4	0.1	0.2
5	0.2	0.1	0.1	0.5	0.1

Experiments on

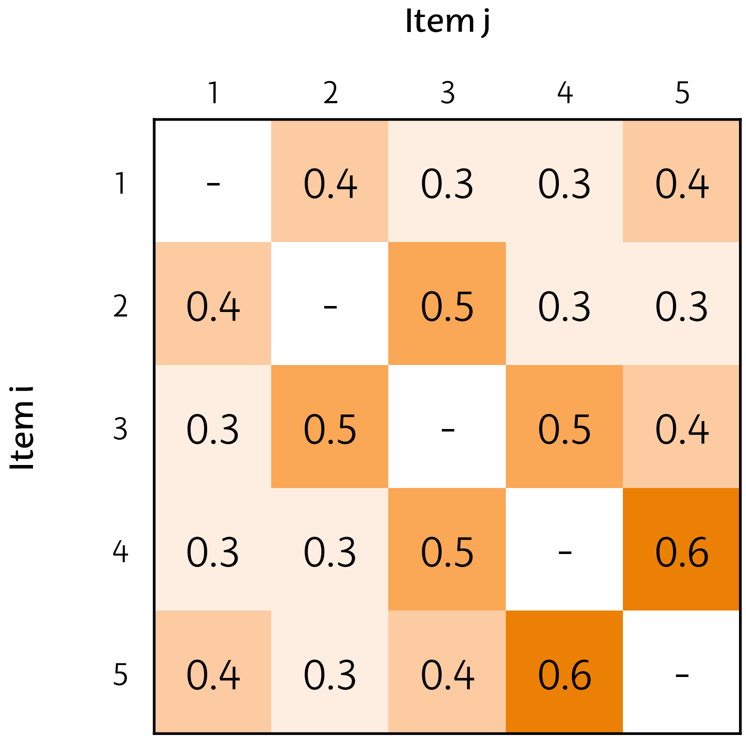
Permutation Flowshop Scheduling Problem

- Node and Edge Histogram-based Sampling Algorithms (EHBSA & NHBSA)
  (Tsutsui et al. 2002, Tsutsui et al. 2006)

Population

54123
42351
12354
24351
31452
23415
23451
25431
12543
53124

Edge Histogram



Item j

Item i	1	2	3	4	5
1	-	0.4	0.3	0.3	0.4
2	0.4	-	0.5	0.3	0.3
3	0.3	0.5	-	0.5	0.4
4	0.3	0.3	0.5	-	0.6
5	0.4	0.3	0.4	0.6	-

The **group of permutations** as a subset of **integers group**

$$n = 3$$

111
112
113
121
122
123
131
132
133
211
212
213
221
222
223
231
232
233
311
312
313
321
322
323
331
332
333

The **group of permutations** as a subset of **integers group**

$n = 3$

111
112
113
121
122
123
131
132
133
211
212
213
221
222
223
231
232
233
311
312
313
321
322
323
331
332
333

Probability Models on Rankings

Bibliography

- M. A. Fligner and J. S. Verducci (**1998**), Multistage Ranking Models, ***Journal of the American Statistical Association***, vol. 83, no. 403, pp. 892-901.
- D. E. Critchlow, M. A. Fligner, and J. S. Verducci (**1991**), Probability Models on Rankings, ***Journal of Mathematical Psychology***, vol. 35, no. 3, pp. 294-318.
- P. Diaconis (**1988**), Group Representations in Probability and Statistics, ***Institute of Mathematical Statistics.***
- M. A. Fligner and J. S. Verducci (**1986**), Distance based Ranking Models, ***Journal of Royal Statistical Society, Series B***, vol. 48, no. 3, pp. 359-369.
- R. L. Plackett (**1975**), The Analysis of Permutations, ***Applied Statistics***, vol. 24, no. 10, pp. 193-202.
- D. R. Luce (**1959**), Individual Choice Behaviour, ***Wiley.***
- R. A. Bradley AND M. E. Terry (**1952**), Rank Analysis of Incomplete Block Designs: I. The Method of Paired Comparisons, ***Biometrika***, vol. 39, no. 3, pp. 324-345.
- L. L. Thurstone (**1927**), A law of comparative judgment, ***Psychological Review***, vol 34, no. 4, pp. 273-286.

Probability Models on Rankings

$$P(\sigma) = \frac{1}{\psi(\theta)} e^{-\theta D(\sigma, \sigma_0)}$$

Mallows

Distance-based

$$P(\sigma) = \frac{1}{\psi(\theta)} e^{-\sum_{j=1}^{n-1} \theta_j S_j(\sigma, \sigma_0)}$$

Generalized Mallows

$$P(\sigma) = \prod_{i=1}^{n-1} \frac{w_{\sigma(i)}}{\sum_{j=i}^{n} w_{\sigma(j)}}$$

Plackett-Luce

Order statistics

$$P(\sigma) = \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \frac{w_{\sigma(i)}}{w_{\sigma(i)} + w_{\sigma(j)}}$$

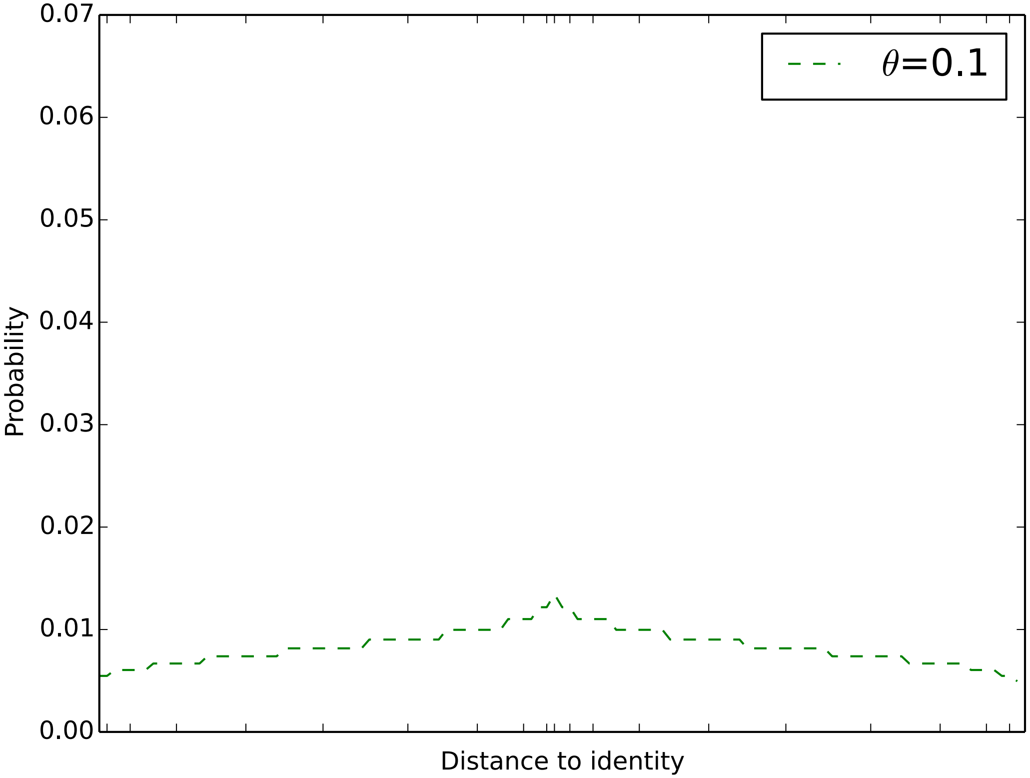
Bradley-Terry

Probability Models on Rankings

The Mallows Model

- A distance-based exponential probability model
  - Central permutation σ_0
  - Spread parameter θ
  - A distance on permutations

$$P(\sigma) = \frac{e^{-\theta D(\sigma,\sigma_0)}}{\psi(\theta)}$$

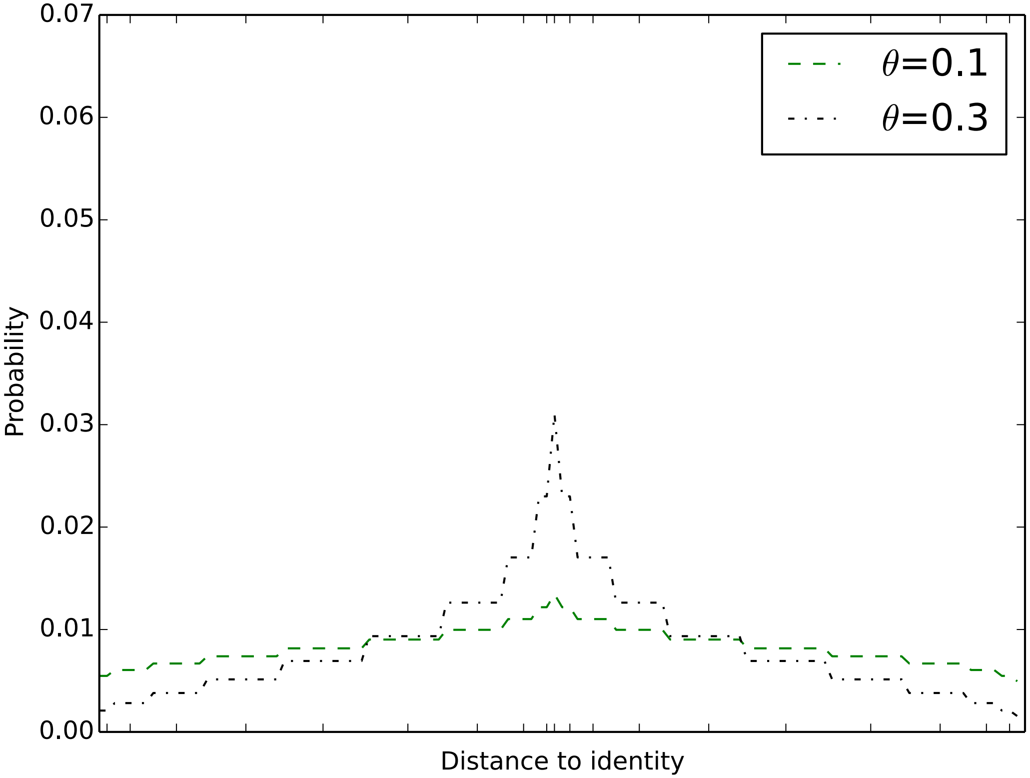


Probability Models on Rankings

The Mallows Model

- A distance-based exponential probability model
  - Central permutation σ_0
  - Spread parameter θ
  - A distance on permutations

$$P(\sigma) = \frac{e^{-\theta D(\sigma,\sigma_0)}}{\psi(\theta)}$$

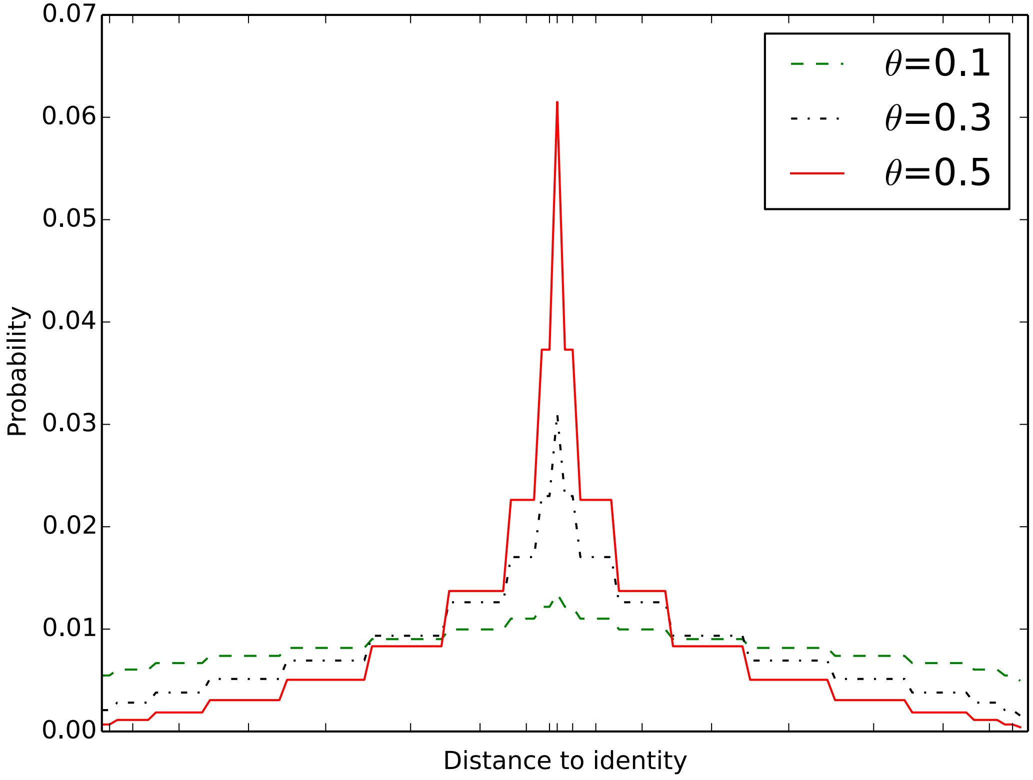


Probability Models on Rankings

The Mallows Model

- A distance-based exponential probability model
  - Central permutation σ_0
  - Spread parameter θ
  - A distance on permutations

$$P(\sigma) = \frac{e^{-\theta D(\sigma,\sigma_0)}}{\psi(\theta)}$$



Probability Models on Rankings

Kendall's-τ distance

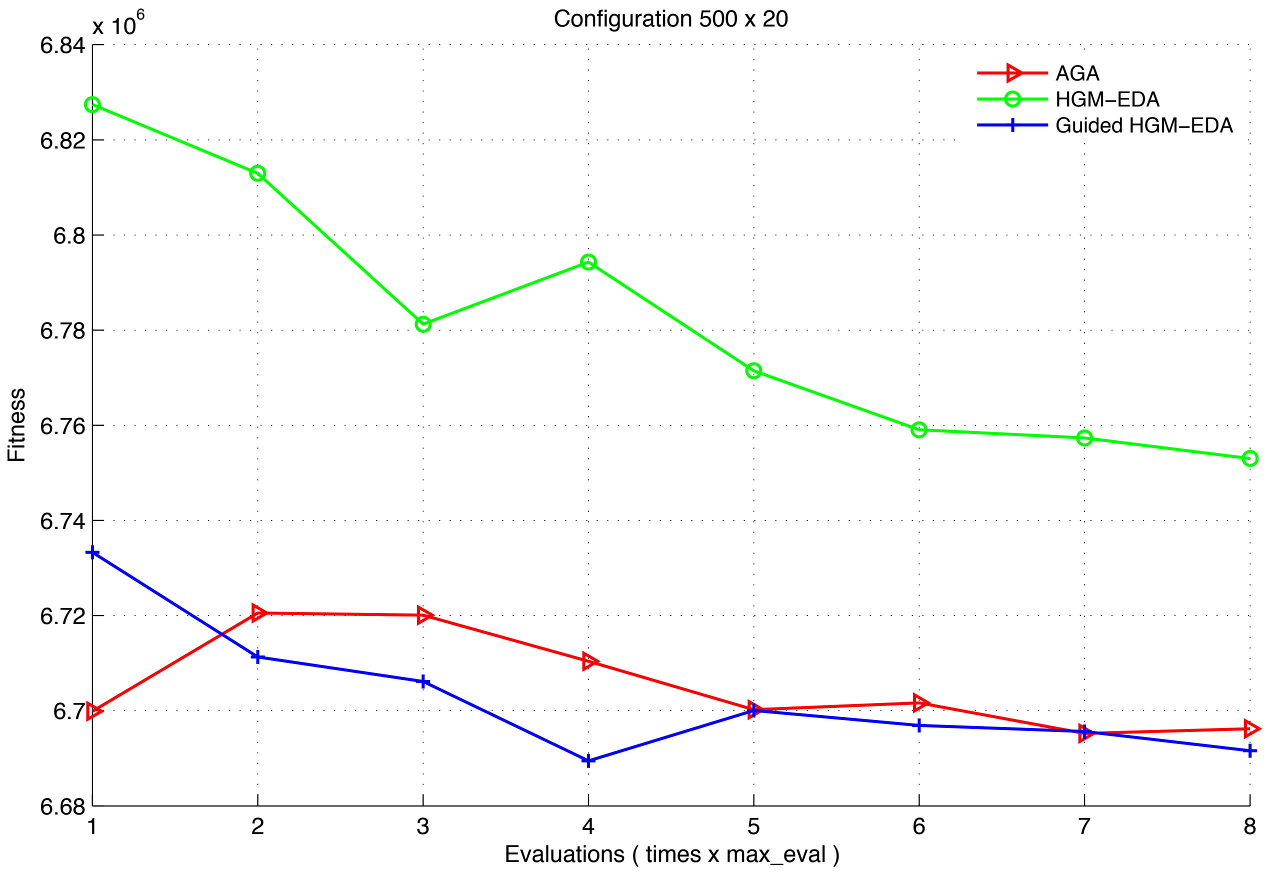
- **Kendall's-τ distance**: calculates the number of pairwise disagreements.

$$\sigma_A = 53412$$
$$\sigma_B = 12345$$

$$D_\tau(\sigma_A, \sigma_B) = ?8$$

	σ_A	σ_B	
1-2	$1 \prec 2$	$1 \prec 2$	
1-3	$3 \prec 1$	$3 \not\prec 1$	←
1-4	$4 \prec 1$	$4 \not\prec 1$	←
1-5	$5 \prec 1$	$5 \not\prec 1$	←
2-3	$3 \prec 2$	$3 \not\prec 2$	←
2-4	$4 \prec 2$	$4 \not\prec 2$	←
2-5	$5 \prec 2$	$5 \not\prec 2$	←
3-4	$3 \prec 4$	$3 \prec 4$	
3-5	$5 \prec 3$	$5 \not\prec 3$	←
4-5	$5 \prec 4$	$5 \not\prec 4$	←

Probability Models on Rankings



Improved state-of-the-art !!!

J. Ceberio et al. (2013) A Distance-based Ranking Model EDA for the PFSP.
IEEE Transactions On Evolutionary Computation , vol 18, No. 2, Pp. 286-300.

Linear Ordering Problem and Neighborhood Topology

Example 2

Linear Ordering Problem (LOP)

0	16	11	15	7
21	0	14	15	9
26	23	0	26	12
22	22	11	0	13
30	28	25	24	0

$$\mathbf{B} = [b_{k,l}]_{5\times 5}$$

Linear Ordering Problem (LOP)

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 16 & 11 & 15 & 7 \\ 2 & 21 & 0 & 14 & 15 & 9 \\ 3 & 26 & 23 & 0 & 26 & 12 \\ 4 & 22 & 22 & 11 & 0 & 13 \\ 5 & 30 & 28 & 25 & 24 & 0 \end{array}$$

$$f(\sigma) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} b_{\sigma(i),\sigma(j)}$$

$\mathbf{B} = [b_{k,l}]_{5\times 5}$

$$\sigma = 12345$$

$$f(\sigma) = 138$$

Linear Ordering Problem (LOP)

$$
\begin{array}{c|ccccc}
 & 5 & 3 & 4 & 2 & 1 \\
\hline
5 & 0 & 25 & 24 & 28 & 30 \\
3 & 12 & 0 & 26 & 23 & 26 \\
4 & 13 & 11 & 0 & 22 & 22 \\
2 & 9 & 14 & 15 & 0 & 21 \\
1 & 7 & 11 & 15 & 16 & 0
\end{array}
$$

$$f(\sigma) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} b_{\sigma(i),\sigma(j)}$$

$\mathbf{B} = [b_{k,l}]_{5\times 5}$

$$\sigma = 53421$$

$$f(\sigma) = 247$$

The insert neighborhood

Moving to Landscape Context...

- Two solutions σ and σ' are neighbors if σ' is obtained by moving an item of σ from position i to position j

1	2	3	4	5

The insert neighborhood

Moving to Landscape Context...

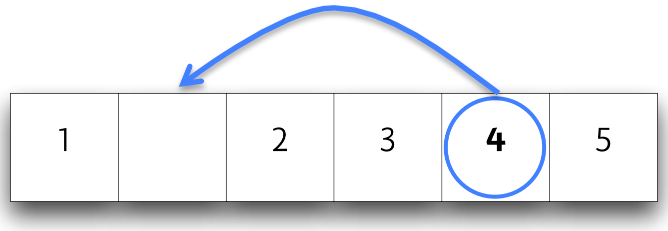
- Two solutions σ and σ' are neighbors if σ' is obtained by moving an item of σ from position i to position j

1	2	3	**4**	5

The insert neighborhood

Moving to Landscape Context...

- Two solutions σ and σ' are neighbors if σ' is obtained by moving an item of σ from position i to position j



The insert neighborhood

Moving to Landscape Context...

- Two solutions σ and σ' are neighbors if σ' is obtained by moving an item of σ from position i to position j

1	**4**	2	3	5

How is the operation translated to the LOP?

Linear Ordering Problem (LOP)

An insert operation...

	1	2	3	4	5
1	0	16	11	15	7
2	21	0	14	15	9
3	26	23	0	26	12
4	22	22	11	0	13
5	30	28	25	24	0

$\mathbf{B} = [b_{k,l}]_{5\times 5}$

Linear Ordering Problem (LOP)

An insert operation...

	1	2	3	4	5
1	0	16	11	**15**	7
2	21	0	14	**15**	9
3	26	23	0	**26**	12
4	**22**	**22**	**11**	**0**	**13**
5	30	28	25	**24**	0

$\mathbf{B} = [b_{k,l}]_{5\times 5}$

Linear Ordering Problem (LOP)

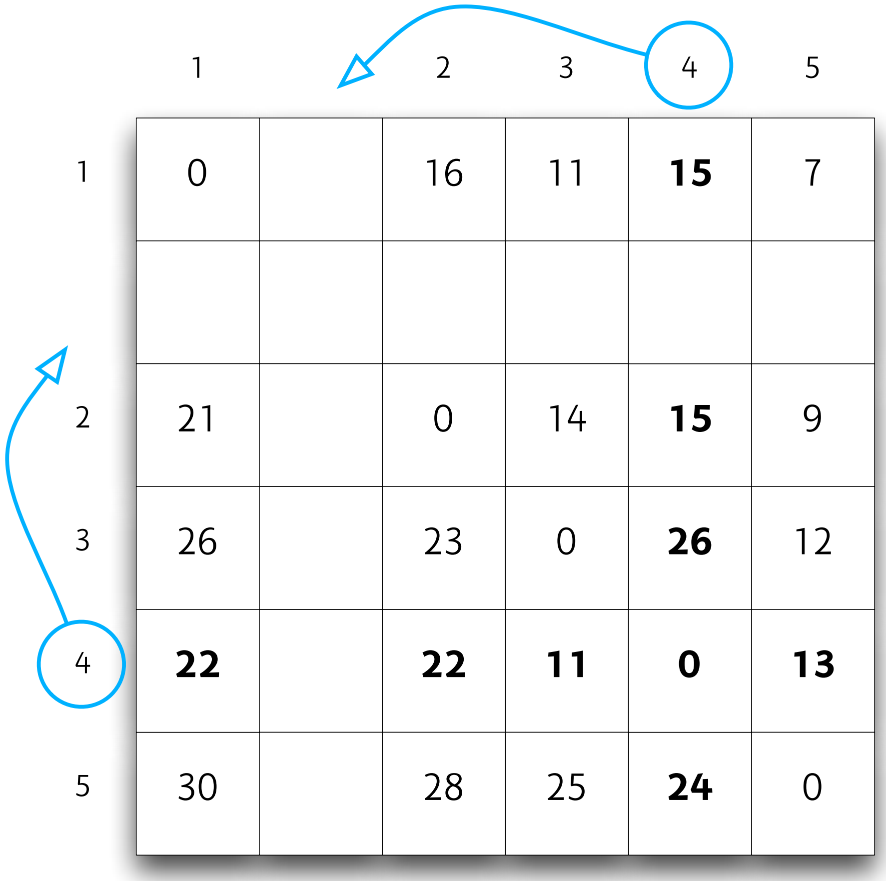
An insert operation...

	1		2	3	4	5
1	0		16	11	**15**	7
2	21		0	14	**15**	9
3	26		23	0	**26**	12
4	**22**		**22**	**11**	**0**	**13**
5	30		28	25	**24**	0

$\mathbf{B} = [b_{k,l}]_{5\times 5}$

Linear Ordering Problem (LOP)

An insert operation...



$$\begin{array}{c|cccccc}
 & 1 & & 2 & 3 & 4 & 5 \\
\hline
1 & 0 & & 16 & 11 & \mathbf{15} & 7 \\
 & & & & & & \\
2 & 21 & & 0 & 14 & \mathbf{15} & 9 \\
3 & 26 & & 23 & 0 & \mathbf{26} & 12 \\
4 & \mathbf{22} & & \mathbf{22} & \mathbf{11} & \mathbf{0} & \mathbf{13} \\
5 & 30 & & 28 & 25 & \mathbf{24} & 0
\end{array}$$

$\mathbf{B} = [b_{k,l}]_{5\times5}$

Linear Ordering Problem (LOP)

An insert operation...

	1	4	2	3	5
1	0	**15**	16	11	7
4	**22**	**0**	**22**	**11**	**13**
2	21	**15**	0	14	9
3	26	**26**	23	0	12
5	30	**24**	28	25	0

$$\mathbf{B} = [b_{k,l}]_{5\times 5}$$

Linear Ordering Problem (LOP)

An insert operation...

Before

	1	2	3	4	5
1	0	16	11	15	7
2	21	0	14	15	9
3	26	23	0	26	12
4	22	22	11	0	13
5	30	28	25	24	0

$$\sigma = 12345$$
$$f(\sigma) = 138$$

After

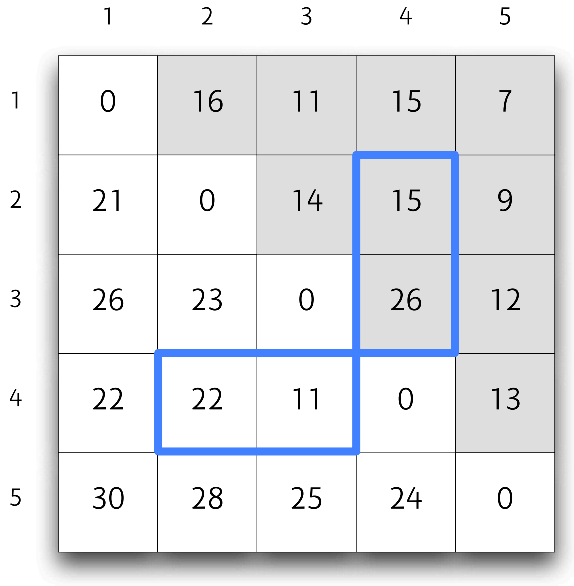
	1	4	2	3	5
1	0	15	16	11	7
4	22	**0**	22	11	13
2	21	15	0	14	9
3	26	26	23	0	12
5	30	24	28	25	0

$$\sigma' = 14235$$
$$f(\sigma) = 130$$

Linear Ordering Problem (LOP)

An insert operation...

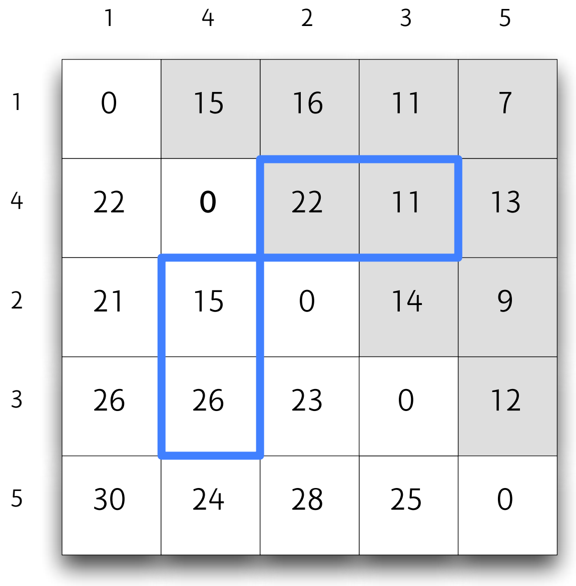
Before



	1	2	3	4	5
1	0	16	11	15	7
2	21	0	14	15	9
3	26	23	0	26	12
4	22	22	11	0	13
5	30	28	25	24	0

$$\sigma = 12345$$
$$f(\sigma) = 138$$

After



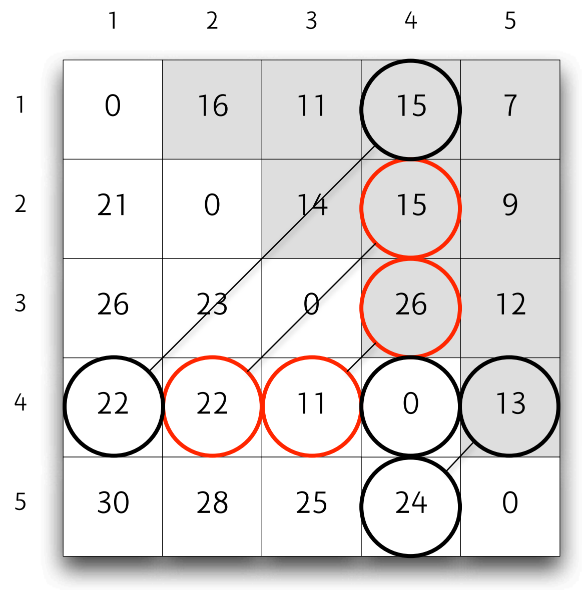
	1	4	2	3	5
1	0	15	16	11	7
4	22	**0**	22	11	13
2	21	15	0	14	9
3	26	26	23	0	12
5	30	24	28	25	0

$$\sigma' = 14235$$
$$f(\sigma) = 130$$

Linear Ordering Problem (LOP)

An insert operation...

Before

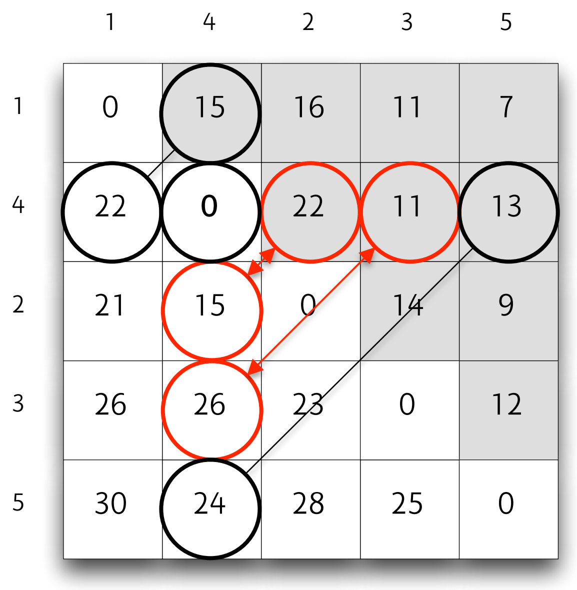


	1	2	3	4	5
1	0	16	11	15	7
2	21	0	14	15	9
3	26	23	0	26	12
4	22	22	11	0	13
5	30	28	25	24	0

$$\sigma = 12345$$

$$f(\sigma) = 138$$

After



	1	4	2	3	5
1	0	15	16	11	7
4	22	0	22	11	13
2	21	15	0	14	9
3	26	26	23	0	12
5	30	24	28	25	0

$$\sigma' = 14235$$

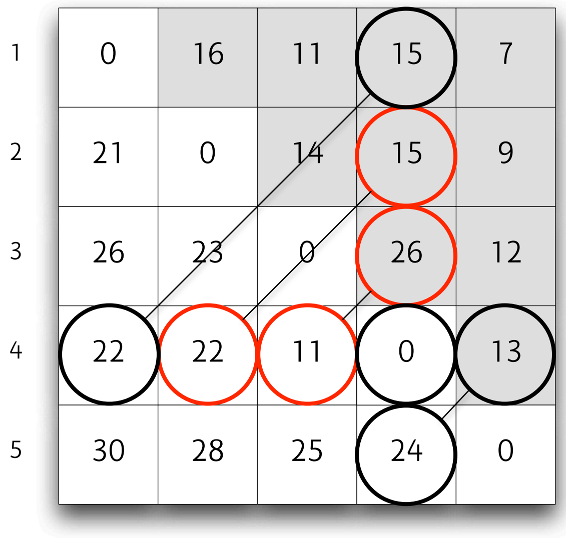
$$f(\sigma) = 130$$

Two pairs of entries associated to the **item 4** exchanged their position.

Linear Ordering Problem (LOP)

An insert operation...

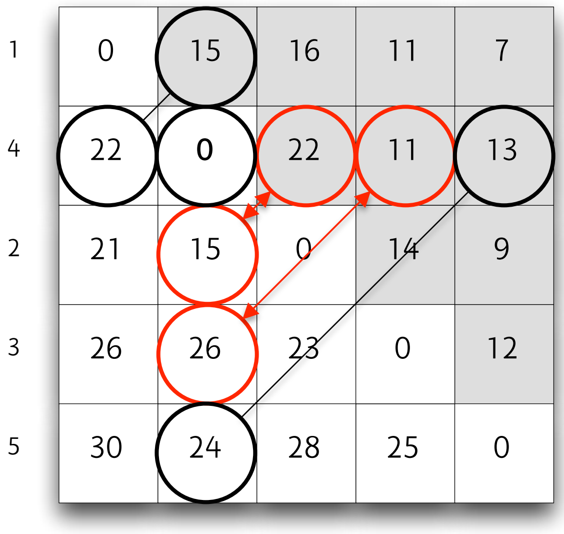
Before



	1	2	3	4	5
1	0	16	11	15	7
2	21	0	14	15	9
3	26	23	0	26	12
4	22	22	11	0	13
5	30	28	25	24	0

$$\sigma = 12345$$
$$f(\sigma) = 138$$

After



	1	4	2	3	5
1	0	15	16	11	7
4	22	0	22	11	13
2	21	15	0	14	9
3	26	26	23	0	12
5	30	24	28	25	0

$$\sigma' = 14235$$
$$f(\sigma) = 130$$

The contribution of the **item 4** to the objective function varied from **69 to 61**.

Linear Ordering Problem (LOP)

The contribution of an item to f

	1	2	3	4	5
1	0	16	11	15	7
2	21	0	14	15	9
3	26	23	0	26	12
4	22	22	11	0	13
5	30	28	25	24	0

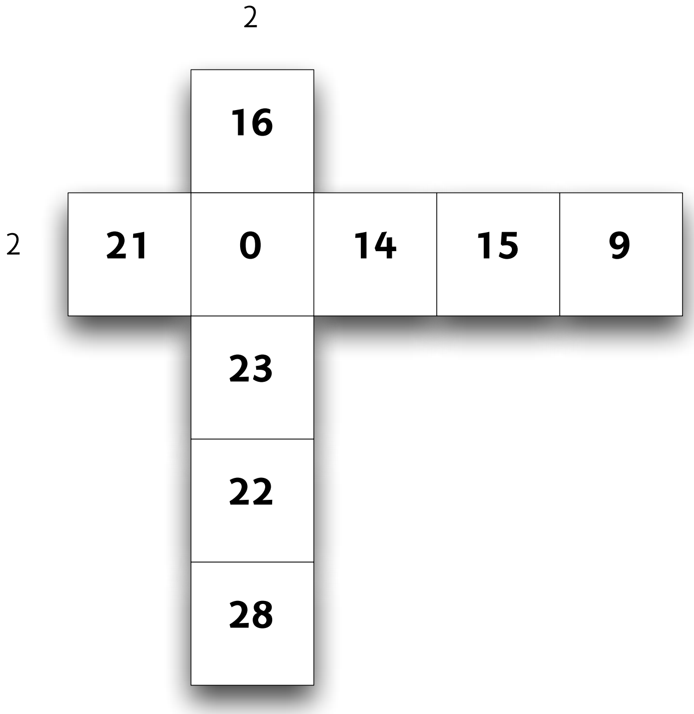
Linear Ordering Problem (LOP)

The contribution of an item to ***f***

	1	2	3	4	5
1	0	**16**	11	15	7
2	**21**	**0**	**14**	**15**	**9**
3	26	**23**	0	26	12
4	22	**22**	11	0	13
5	30	**28**	25	24	0

Linear Ordering Problem (LOP)

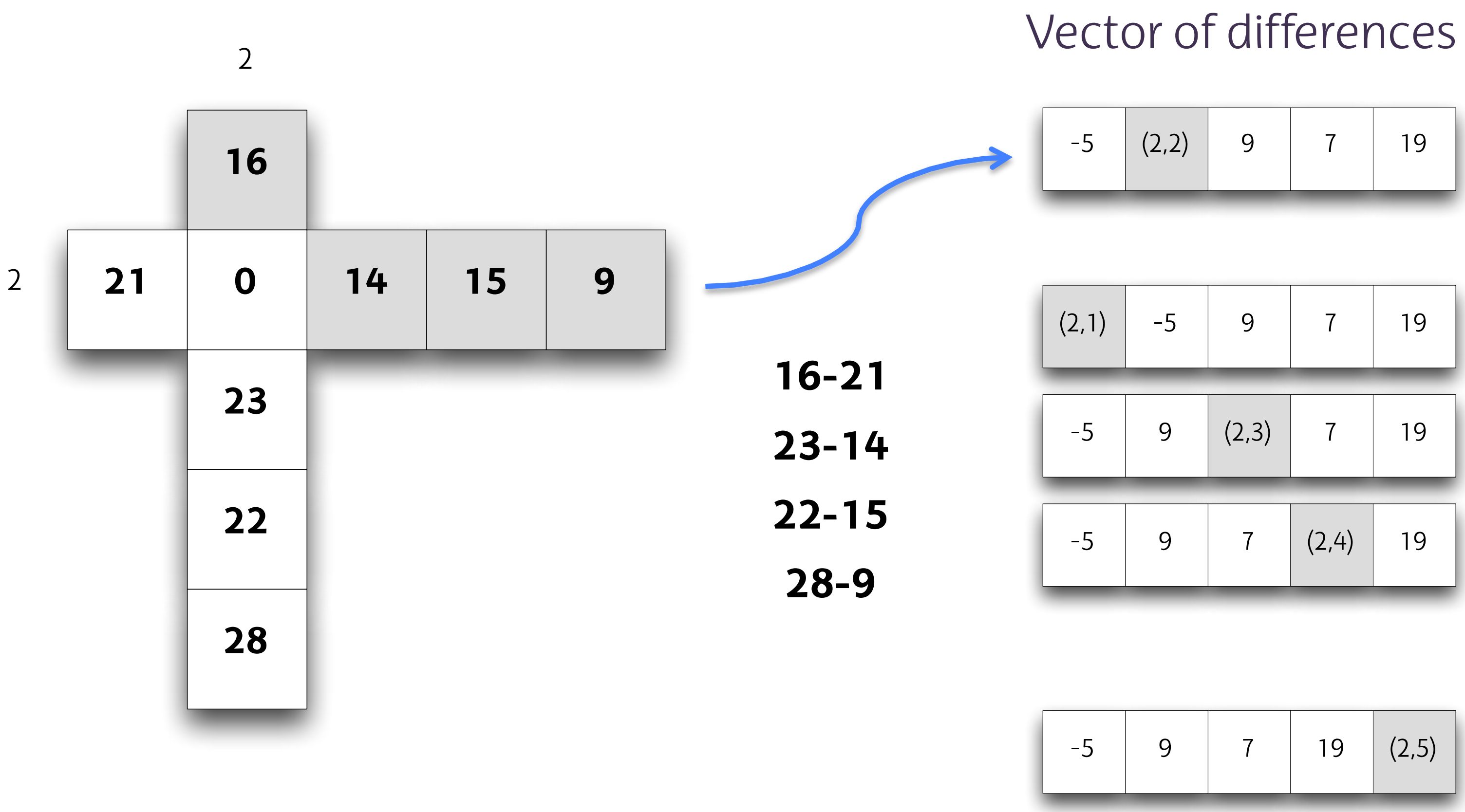
The contribution of an item to *f*



		2			
		16			
2	21	0	14	15	9
		23			
		22			
		28			

Linear Ordering Problem (LOP)

The contribution of an item to *f*



16-21

23-14

22-15

28-9

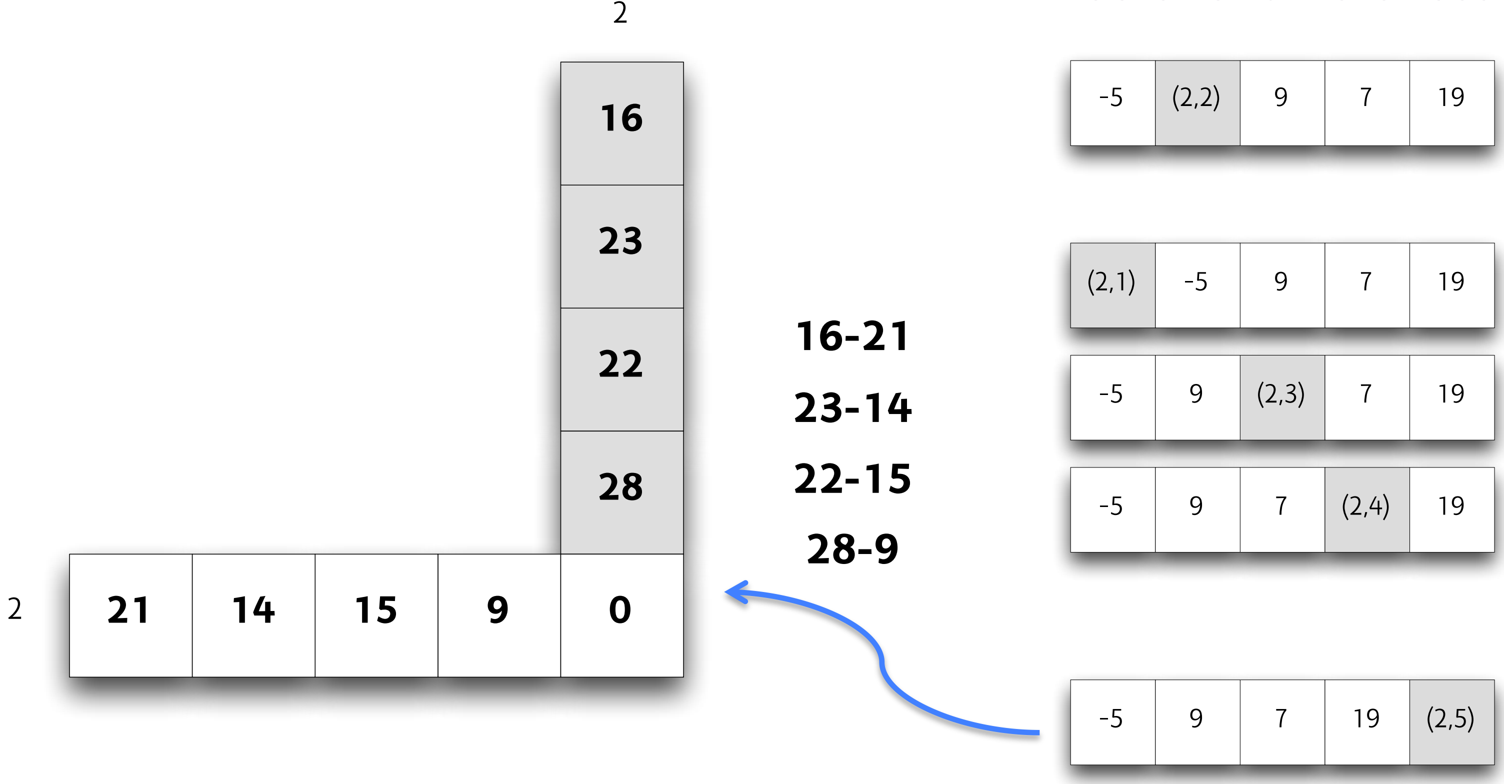
Vector of differences

-5	(2,2)	9	7	19
(2,1)	-5	9	7	19
-5	9	(2,3)	7	19
-5	9	7	(2,4)	19
-5	9	7	19	(2,5)

Contribution: 54

Linear Ordering Problem (LOP)

The contribution of an item to *f*



2

16
23
22
28

2	**21**	**14**	**15**	**9**	**0**

16-21

23-14

22-15

28-9

Vector of differences

-5	(2,2)	9	7	19
(2,1)	-5	9	7	19
-5	9	(2,3)	7	19
-5	9	7	(2,4)	19
-5	9	7	19	(2,5)

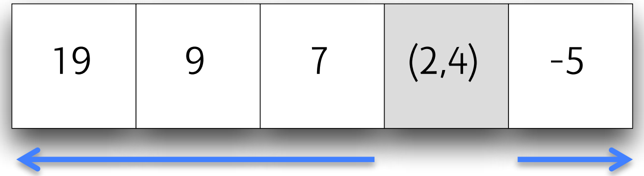
Contribution: 89

The vector of differences

Local optima

What happens in local optimal solutions?

There is no movement that improves the contribution of any item



19	9	7	(2,4)	-5

All the partial sums of differences to the **left** must be **positive**

$7 > 0$

$9 + 7 > 0$

$19 + 9 + 7 > 0$

$0 < -5$

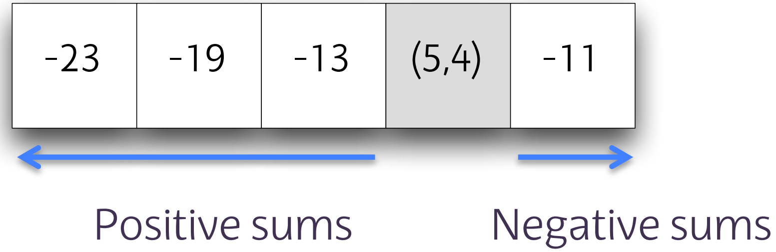
All the partial sums of differences to the **right** must be **negative**

Depends on the overall solution

The vector of differences

Local optima

But,



-23	-19	-13	(5,4)	-11

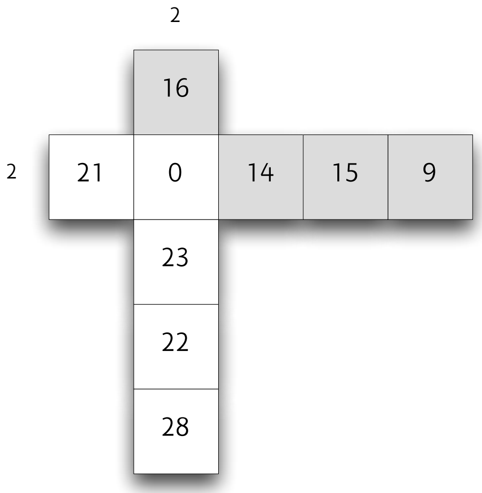
Positive sums

Negative sums

In order to produce local optima,
item 5 must be placed in the **first position**

The restrictions matrix

We propose an algorithm to calculate the restricted positions of the items:



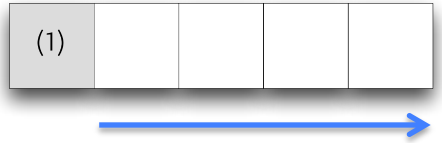
1. Vector of differences.

-5	(2,2)	9	7	19

2. Sort differences

19	9	7	-5

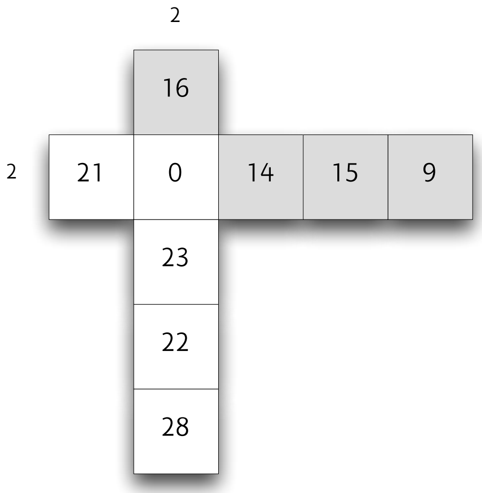
3. Study the most favorable ordering of differences in each positions



All the partial sums of differences to the **right** must be **negative**

The restrictions matrix

We propose an algorithm to calculate the restricted positions of the items:



		2			
		16			
2	21	0	14	15	9
		23			
		22			
		28			

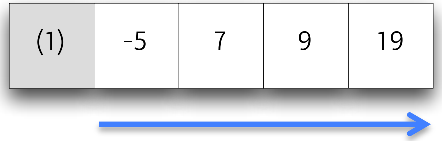
1. Vector of differences.

-5	(2,2)	9	7	19

2. Sort differences

19	9	7	-5

3. Study the most favorable ordering of differences in each positions

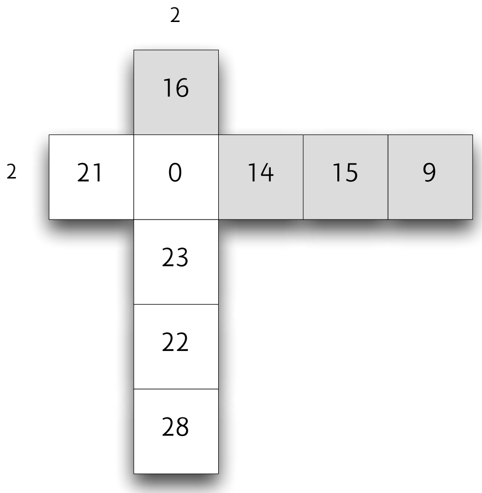


(1)	-5	7	9	19

All the partial sums of differences to the **right** must be **negative**

The restrictions matrix

We propose an algorithm to calculate the restricted positions of the items:



1. Vector of differences.

-5	(2,2)	9	7	19

2. Sort differences

19	9	7	-5

3. Study the most favorable ordering of differences in each positions

(1)	-5	7	9	19
19	(2)	-5	7	9
9	19	(3)	-5	7
7	9	19	(4)	-5
-5	7	9	19	(5)

Non-local optima (rows 1–3)

Possible local optima (rows 4–5)

The restrictions matrix

	1	2	3	4	5
1	0	16	11	15	7
2	21	0	14	15	9
3	26	23	0	26	12
4	22	22	11	0	13
5	30	28	25	24	0

$\mathbf{B}$

	1	2	3	4	5
1	0	0	0	0	1
2	0	0	0	1	1
3	1	1	0	0	0
4	0	1	1	1	1
5	1	0	0	0	0

$\mathbf{R}$

Time complexity: $O(n^3)$

The restricted insert neighborhood

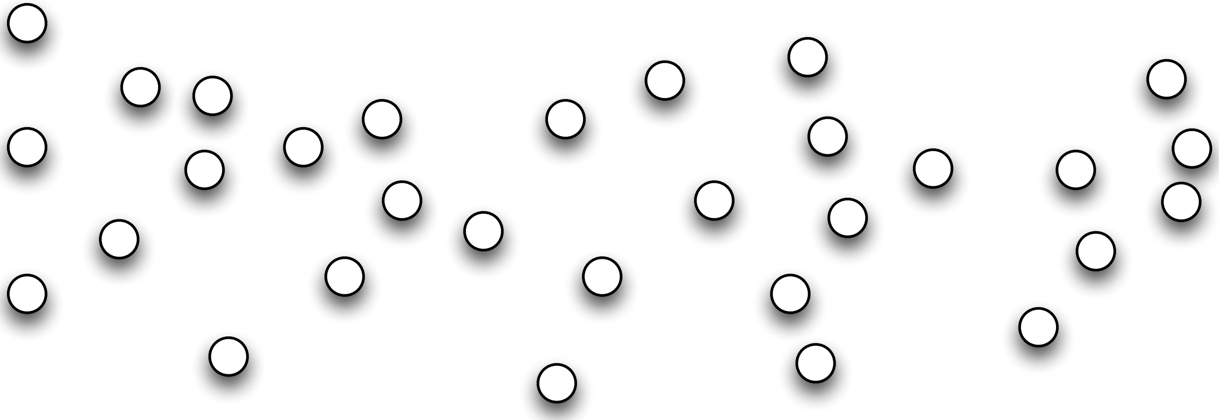
- Incorporate the restrictions matrix to the insert neighborhood.
- Discard the insert operations that move items to the restricted positions.

Theorem

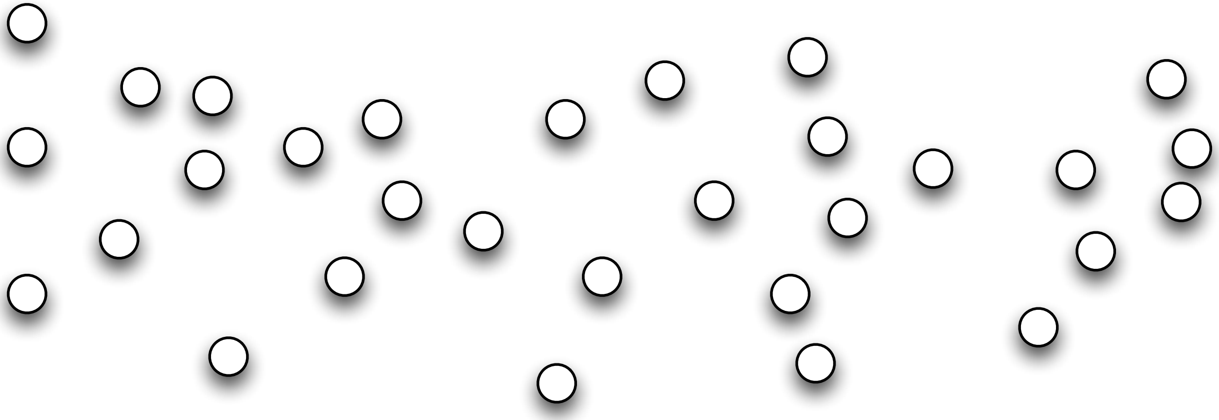
Given a non local optima solution σ*, for every item* $\sigma(i)$*,* $i = 1, \ldots, n$*, the insert movement that maximises its contribution to the fitness function is not given in a restricted position*

The restricted insert neighborhood

Insert neighborhood

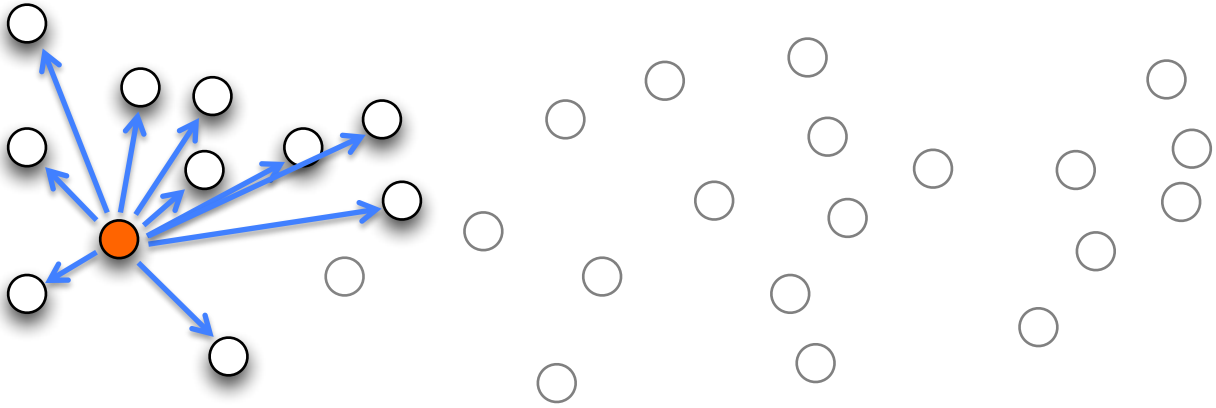


Restricted Insert neighborhood



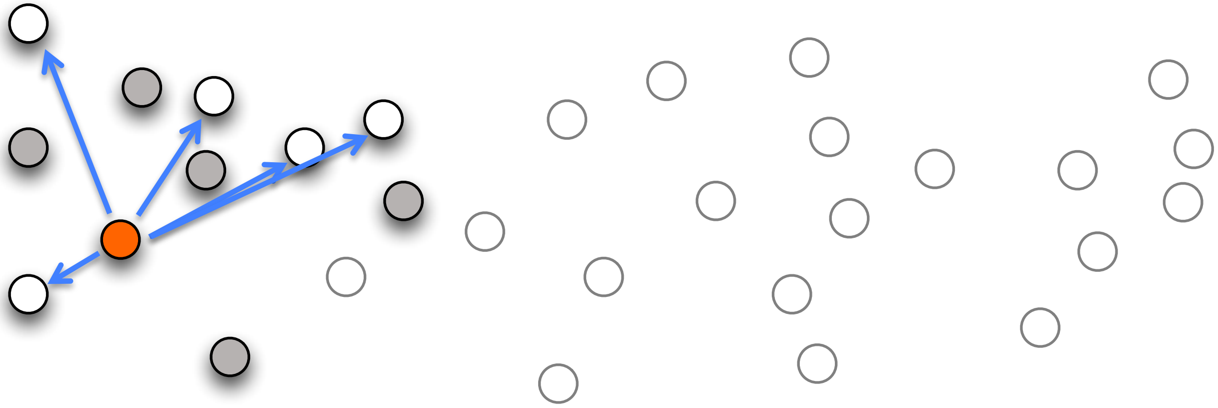
The restricted insert neighborhood

Insert
neighborhood



Evaluations: 10

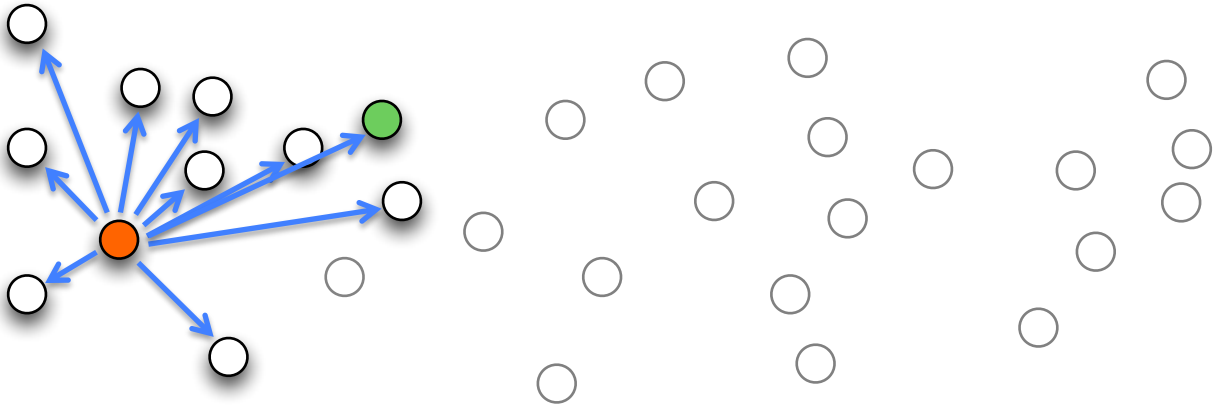
Restricted Insert
neighborhood



Evaluations: 5

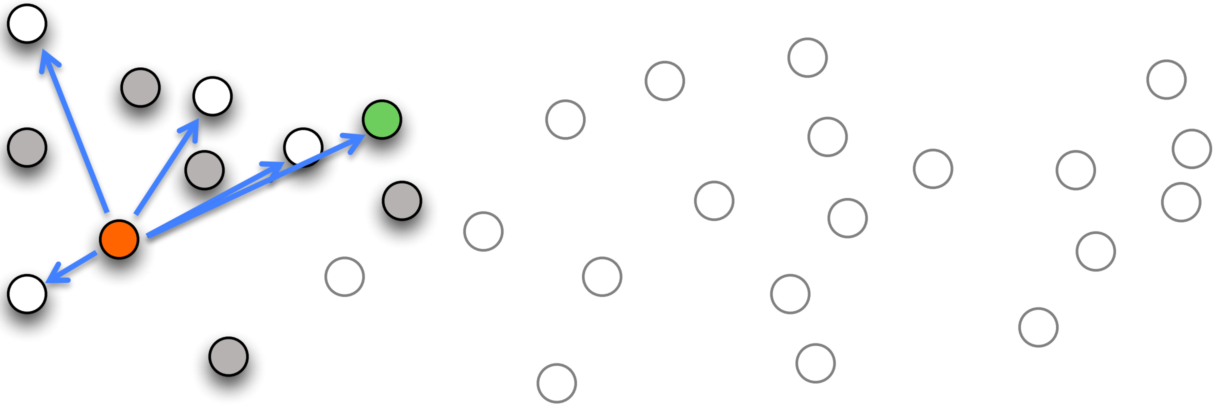
The restricted insert neighborhood

Insert
neighborhood



Evaluations: 10

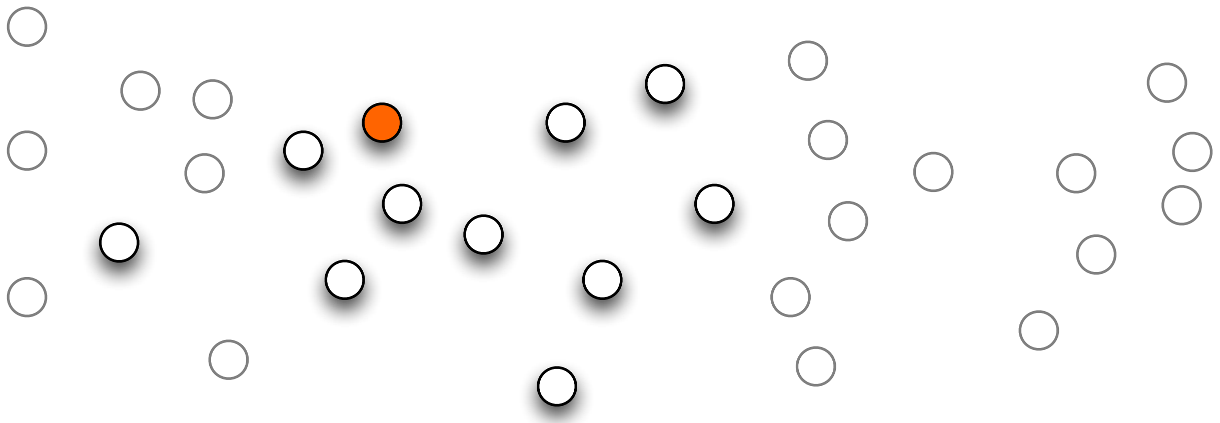
Restricted Insert
neighborhood



Evaluations: 5

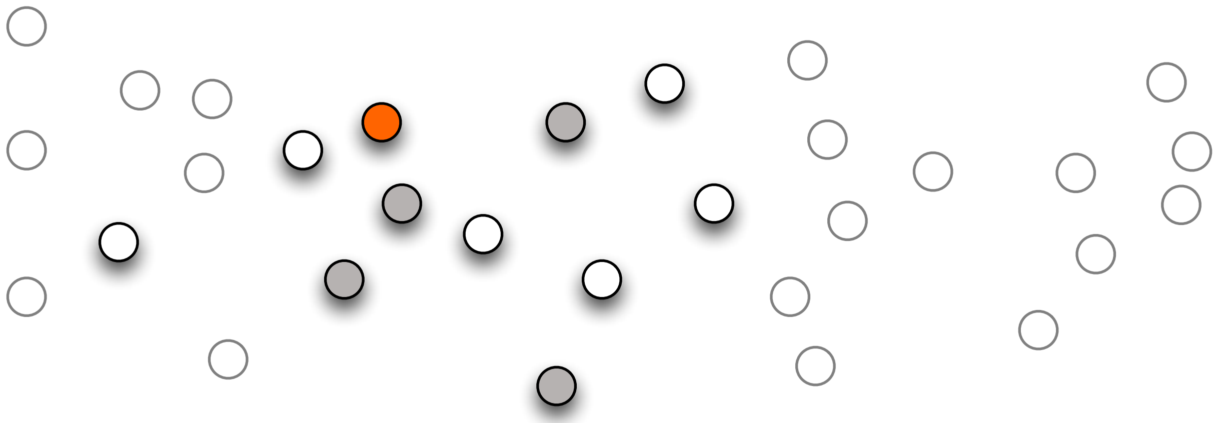
The restricted insert neighborhood

Insert neighborhood



Evaluations: 10

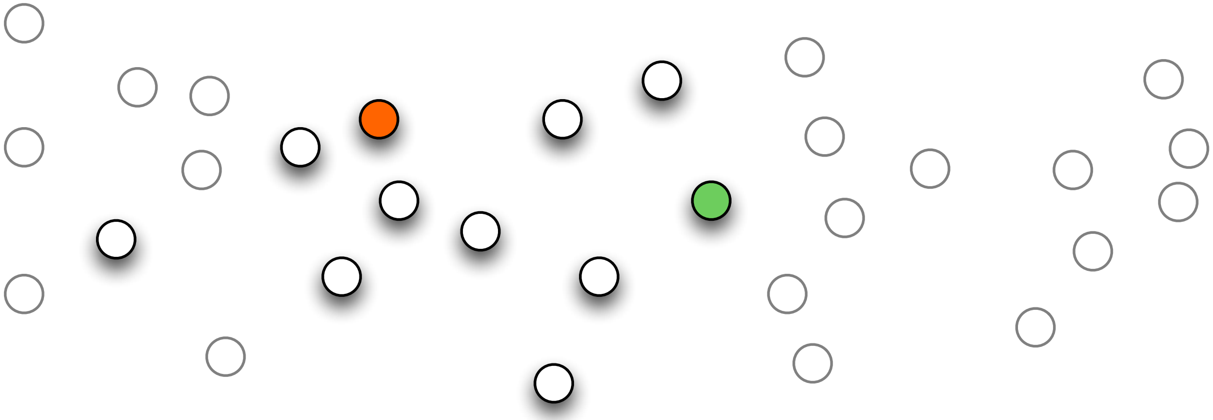
Restricted Insert neighborhood



Evaluations: 5

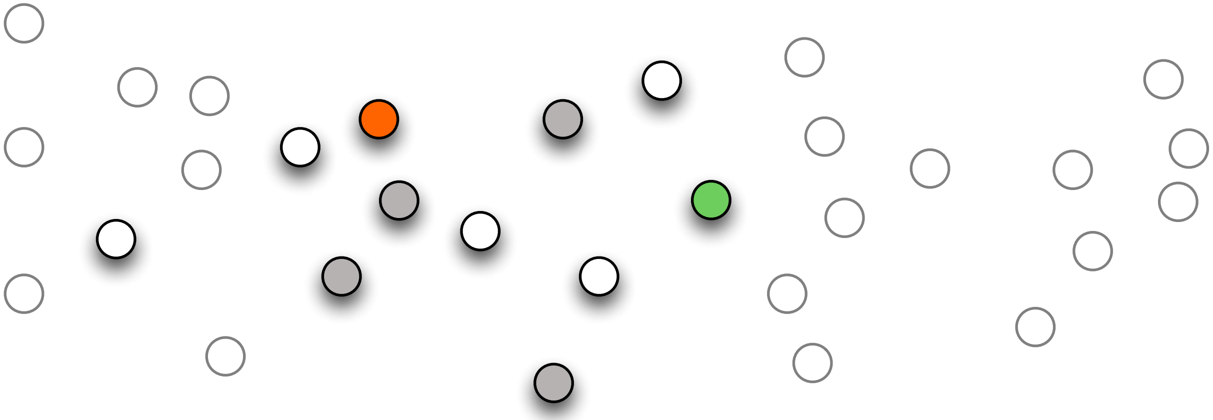
The restricted insert neighborhood

Insert
neighborhood



Evaluations: 20

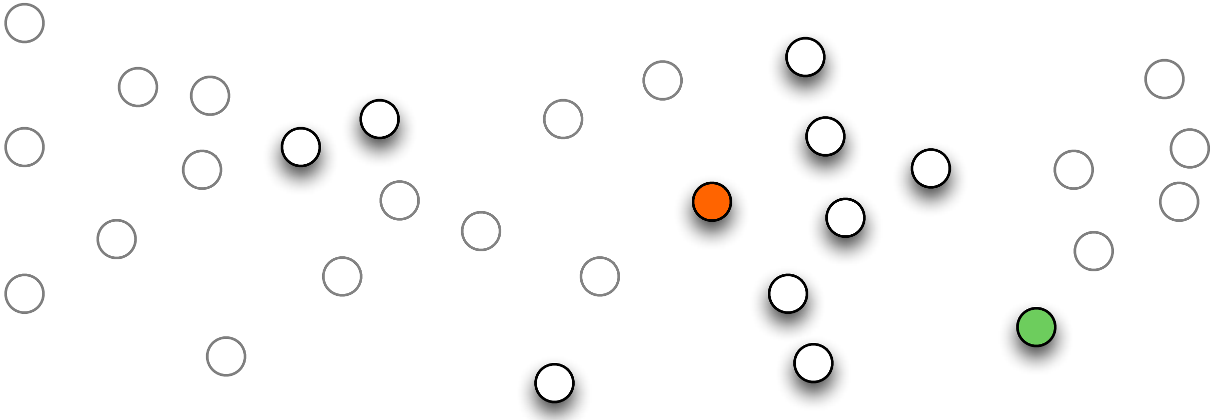
Restricted Insert
neighborhood



Evaluations: 11

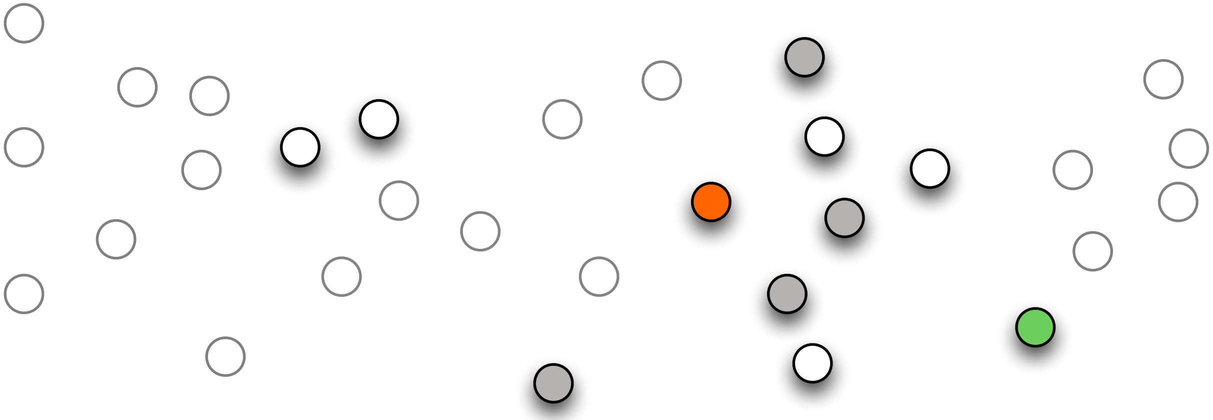
The restricted insert neighborhood

Insert neighborhood



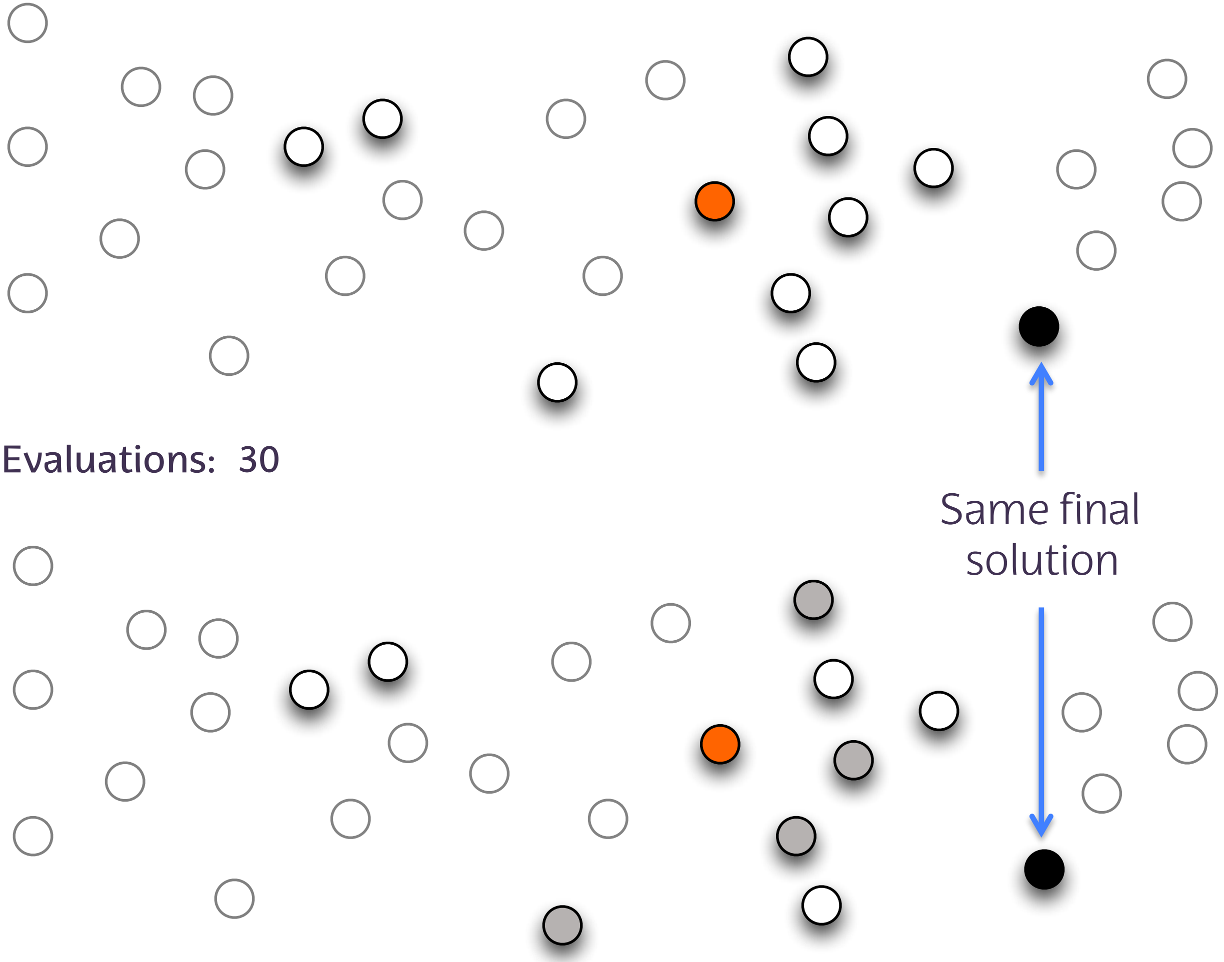
Evaluations: 30

Restricted Insert neighborhood



Evaluations: 17

The restricted insert neighborhood



Insert
neighborhood

Evaluations: 30

Same final
solution

Restricted Insert
neighborhood

Evaluations: 17

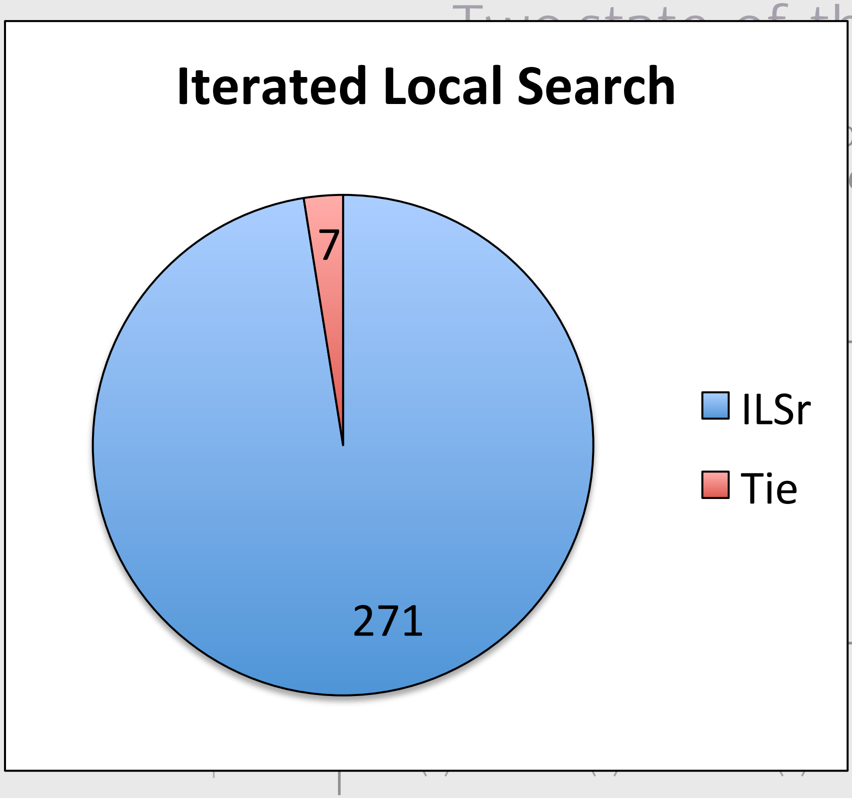
J. Ceberio et al. (2014) The Linear Ordering Problem Revisited. ***European Journal of Operational Research***.

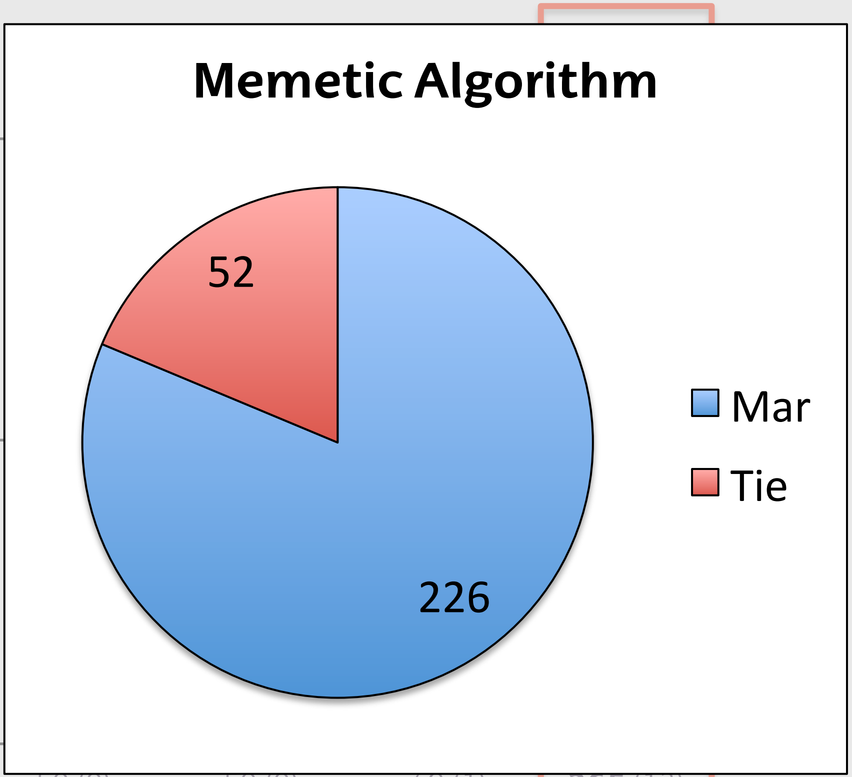
Experiments

Two state-of-the-art algorithms

ordering problem: instances, search space analysis

odelling and Algorithms.





10000n^2 evals.	150	250	300				
MA_r vs MA	39 (0)	34 (5)	43 (7)	50 (0)	50 (0)	49 (1)	**265** (13)
ILS_r vs ILS	33 (6)	37 (2)	46 (4)	42 (8)	43 (7)	45 (5)	**246** (32)

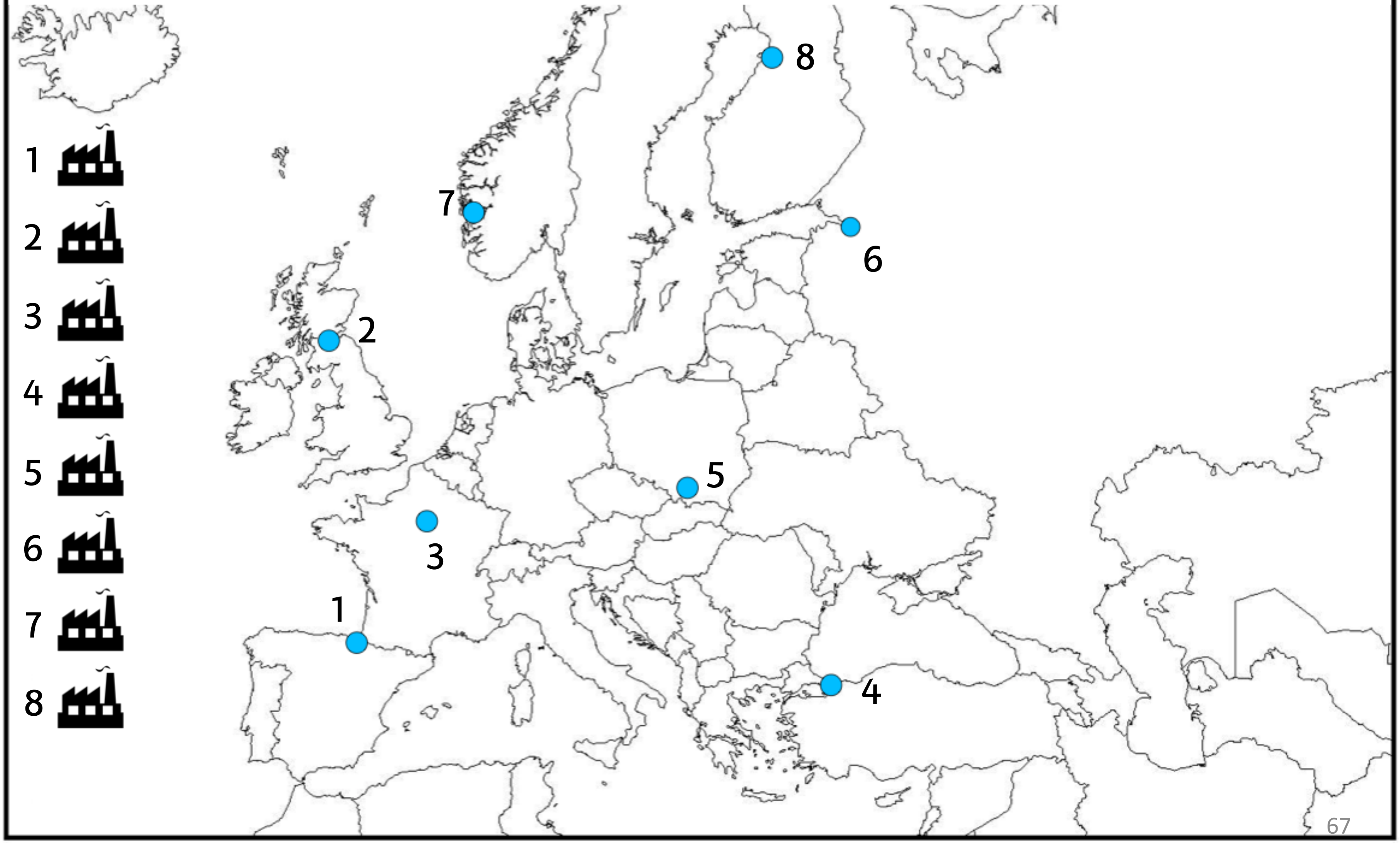
278 instances

Quadratic Assignment Problem and Elementary Landscapes

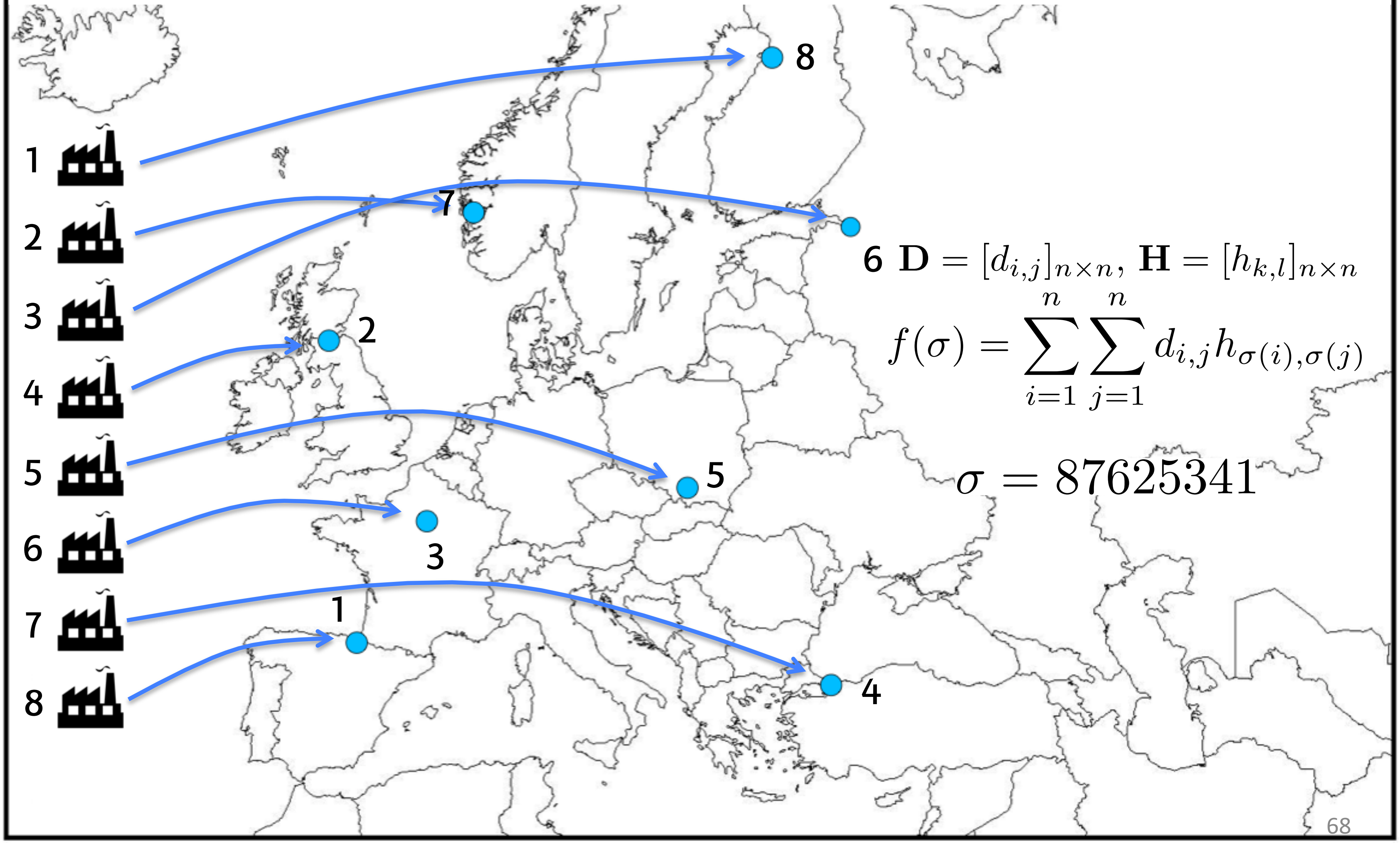
Example 3

Quadratic Assignment Problem (QAP)

1 🏭
2 🏭
3 🏭
4 🏭
5 🏭
6 🏭
7 🏭
8 🏭



Quadratic Assignment Problem (QAP)



$$\mathbf{D} = [d_{i,j}]_{n\times n},\ \mathbf{H} = [h_{k,l}]_{n\times n}$$

$$f(\sigma) = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{i,j} h_{\sigma(i),\sigma(j)}$$

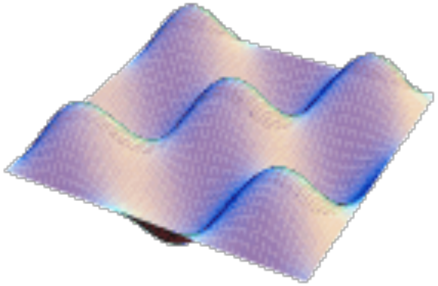
$$\sigma = 87625341$$

Elementary landscapes

Definitions

A landscape is

$$(\mathbb{S}_n, f, N)$$



An elementary landscape fulfills

$$\underset{\pi \in N(\sigma)}{avg} \{f(\pi)\} = f(\sigma) + \frac{k}{|N(\sigma)|}(\bar{f} - f(\sigma))$$

Groover's wave equation

Elementary landscape decomposition

… of the QAP

According to **Chicano et al. 2010**

If the neighborhood N is

Regular

$$|N(\sigma)| = d > 0 \text{ for all } \sigma \in \mathbb{S}_n$$

Symmetric

$$\text{for all } \sigma, \pi \in \mathbb{S}_n, \pi \in N(\sigma) \iff \sigma \in N(\pi)$$

then the landscape can be decomposed as a sum of elementary landscapes

Elementary landscape decomposition

… of the QAP

According to **Chicano et al. 2010**

QAP

$$f(\sigma) = \sum_{i=1}^{n} \sum_{j=1}^{n} d_{i,j} h_{\sigma(i),\sigma(j)}$$

Generalized QAP

$$g(\sigma) = \sum_{i,j,p,q=1}^{n} \psi_{ijpq} \varphi_{(i,j)(p,q)}(\sigma)$$

Elementary landscape decomposition

... of the QAP

According to **Chicano et al. 2010**

Generalized QAP

$$g(\sigma) = \sum_{i,j,p,q=1}^{n} \psi_{ijpq} \varphi_{(i,j)(p,q)}(\sigma)$$

Under the interchange neighborhood

$$g(\sigma) = \sum_{\substack{i,j,p,q=1 \\ i \neq j \\ p \neq q}}^{n} \psi_{ijpq} \left(\frac{\Omega^1_{(i,j)(p,q)}(\sigma)}{2n} + \frac{\Omega^2_{(i,j)(p,q)}(\sigma)}{2(n-2)} + \frac{\Omega^3_{(i,j)(p,q)}(\sigma)}{n(n-2)} \right)$$

Landscape 1 Landscape 2 Landscape 3

Elementary landscape decomposition

... of the QAP

$$g(\sigma) = \sum_{\substack{i,j,p,q=1 \\ i \neq j \\ p \neq q}}^{n} \psi_{ijpq} \left(\frac{\Omega^1_{(i,j)(p,q)}(\sigma)}{2n} + \frac{\Omega^2_{(i,j)(p,q)}(\sigma)}{2(n-2)} + \frac{\Omega^3_{(i,j)(p,q)}(\sigma)}{n(n-2)} \right)$$

Landscape 1 Landscape 2 Landscape 3

In the classic QAP the matrix $\mathbf{D} = [d_{i,j}]_{n \times n}$ is symmetric, as a result

$$f(\sigma) = \lambda + \sum_{\substack{i,j,p,q=1 \\ i \neq j \\ p \neq q}}^{n} \psi_{ijpq} \frac{\Omega^2_{(i,j)(p,q)}(\sigma)}{2(n-2)} + \sum_{\substack{i,j,p,q=1 \\ i \neq j \\ p \neq q}}^{n} \psi_{ijpq} \frac{\Omega^3_{(i,j)(p,q)}(\sigma)}{n(n-2)}$$

Decomposed QAP

Multi-objectivization

... of the QAP

Single-objective
Problem

$$\sigma^* = \arg\max_{\sigma \in \mathbb{S}_n} f(\sigma)$$

Multi-objective
Problem

$$\text{maximize } F(\sigma), \sigma \in \mathbb{S}_n$$

$$\text{where}$$

$$F(\sigma) = [f_1(\sigma), \ldots, f_m(\sigma)]$$

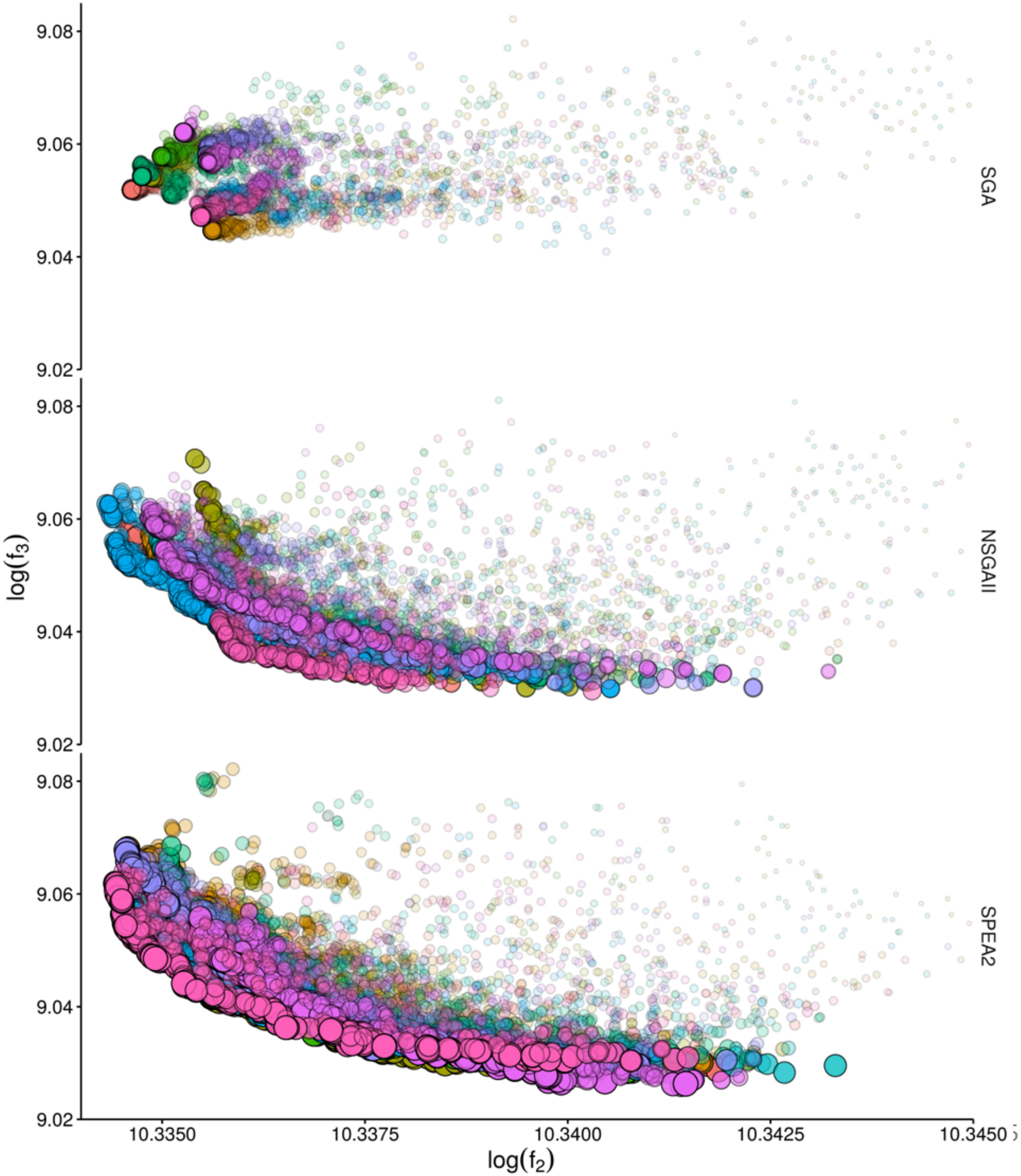
$$f(\sigma) = \lambda + \sum_{\substack{i,j,p,q=1 \\ i \neq j \\ p \neq q}}^{n} \psi_{ijpq} \frac{\Omega^2_{(i,j)(p,q)}(\sigma)}{2(n-2)} + \sum_{\substack{i,j,p,q=1 \\ i \neq j \\ p \neq q}}^{n} \psi_{ijpq} \frac{\Omega^3_{(i,j)(p,q)}(\sigma)}{n(n-2)}$$

Decomposed QAP

Algorithms:

- SGA
- NSGA2
- SPEA2

Movies !!

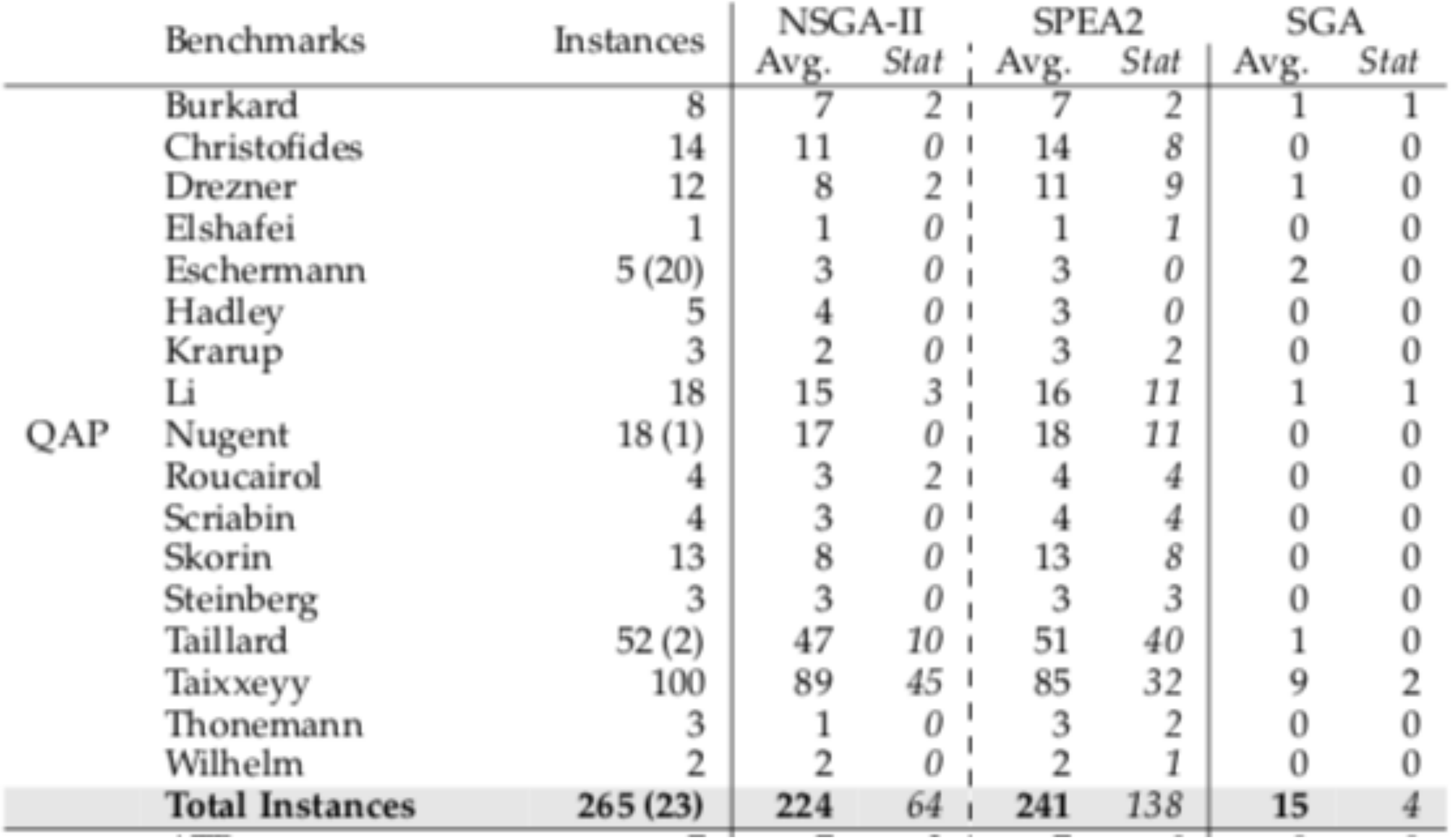


J. Ceberio et al. (2018) Multi-objectivising Combinatorial Optimization Problems by means of Elementary Landscape Decomposition. ***Evolutionary Computation***.

Algorithms:

- SGA
- NSGA2
- SPEA2

Movies !!

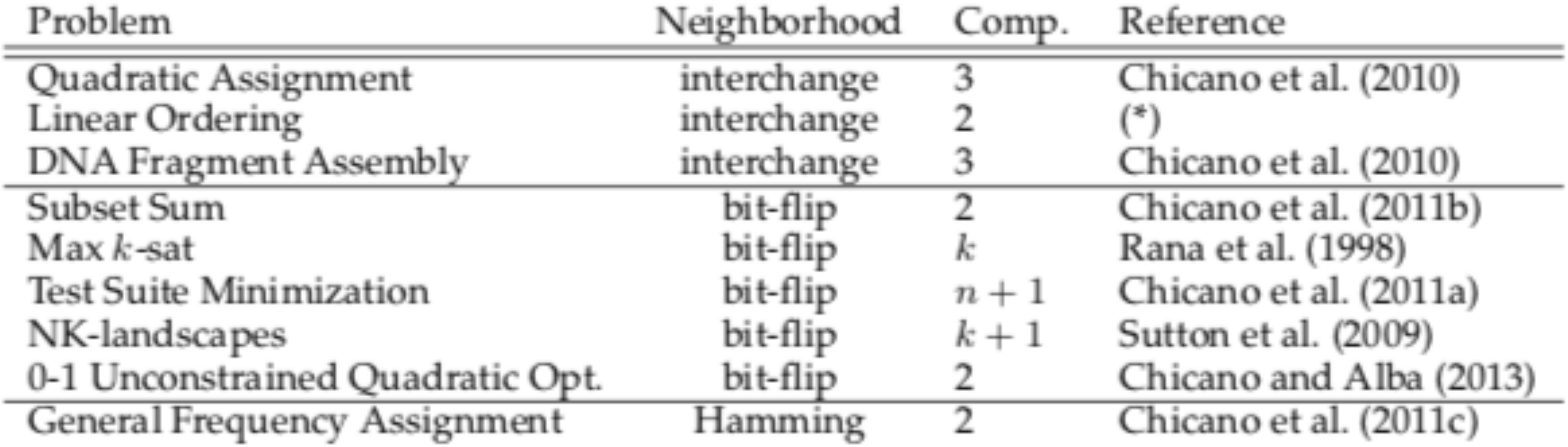


	Benchmarks	Instances	NSGA-II		SPEA2		SGA	
			Avg.	*Stat*	Avg.	*Stat*	Avg.	*Stat*
QAP	Burkard	8	7	*2*	7	*2*	1	1
	Christofides	14	11	*0*	14	*8*	0	0
	Drezner	12	8	*2*	11	*9*	1	0
	Elshafei	1	1	*0*	1	*1*	0	0
	Eschermann	5 (20)	3	*0*	3	*0*	2	0
	Hadley	5	4	*0*	3	*0*	0	0
	Krarup	3	2	*0*	3	*2*	0	0
	Li	18	15	*3*	16	*11*	1	1
	Nugent	18 (1)	17	*0*	18	*11*	0	0
	Roucairol	4	3	*2*	4	*4*	0	0
	Scriabin	4	3	*0*	4	*4*	0	0
	Skorin	13	8	*0*	13	*8*	0	0
	Steinberg	3	3	*0*	3	*3*	0	0
	Taillard	52 (2)	47	*10*	51	*40*	1	0
	Taixxeyy	100	89	*45*	85	*32*	9	2
	Thonemann	3	1	*0*	3	*2*	0	0
	Wilhelm	2	2	*0*	2	*1*	0	0
	Total Instances	**265 (23)**	**224**	*64*	**241**	*138*	**15**	*4*

J. Ceberio et al. (2018) Multi-objectivising Combinatorial Optimization Problems by means of Elementary Landscape Decomposition. ***Evolutionary Computation***.

Multi-objectivization

... of other problems?



Problem	Neighborhood	Comp.	Reference
Quadratic Assignment	interchange	3	Chicano et al. (2010)
Linear Ordering	interchange	2	(*)
DNA Fragment Assembly	interchange	3	Chicano et al. (2010)
Subset Sum	bit-flip	2	Chicano et al. (2011b)
Max k-sat	bit-flip	k	Rana et al. (1998)
Test Suite Minimization	bit-flip	$n+1$	Chicano et al. (2011a)
NK-landscapes	bit-flip	$k+1$	Sutton et al. (2009)
0-1 Unconstrained Quadratic Opt.	bit-flip	2	Chicano and Alba (2013)
General Frequency Assignment	Hamming	2	Chicano et al. (2011c)

Future Research Possibilities

Other Considerations

Possible Future Lines

Benchmarking and difficulty:

- Random generation of instances
- Difficulty of instances.
- Distribution of instances.

Possible codifications:

- Vector of integers
- Vector of continuous values
- Matrices
- Cycles

Permutation Problems

Problem Types:

- Problems with constrains
- Multi-objective
- Deceptive problems
- Dynamic Problems

Non-Standard Permutation Problems:

- Partially Permutation Problems
- Quasi Permutation Problems
- Multi Permutation Problems

Are We Generating Instances Uniformly At Random?

Instances as Rankings of Solutions: Linear Ordering Problem

$$f(\sigma) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} b_{\sigma(i),\sigma(j)}$$

$$\sigma \in \mathbb{S}_n$$

$n = 3$

123

213

132

231

312

321

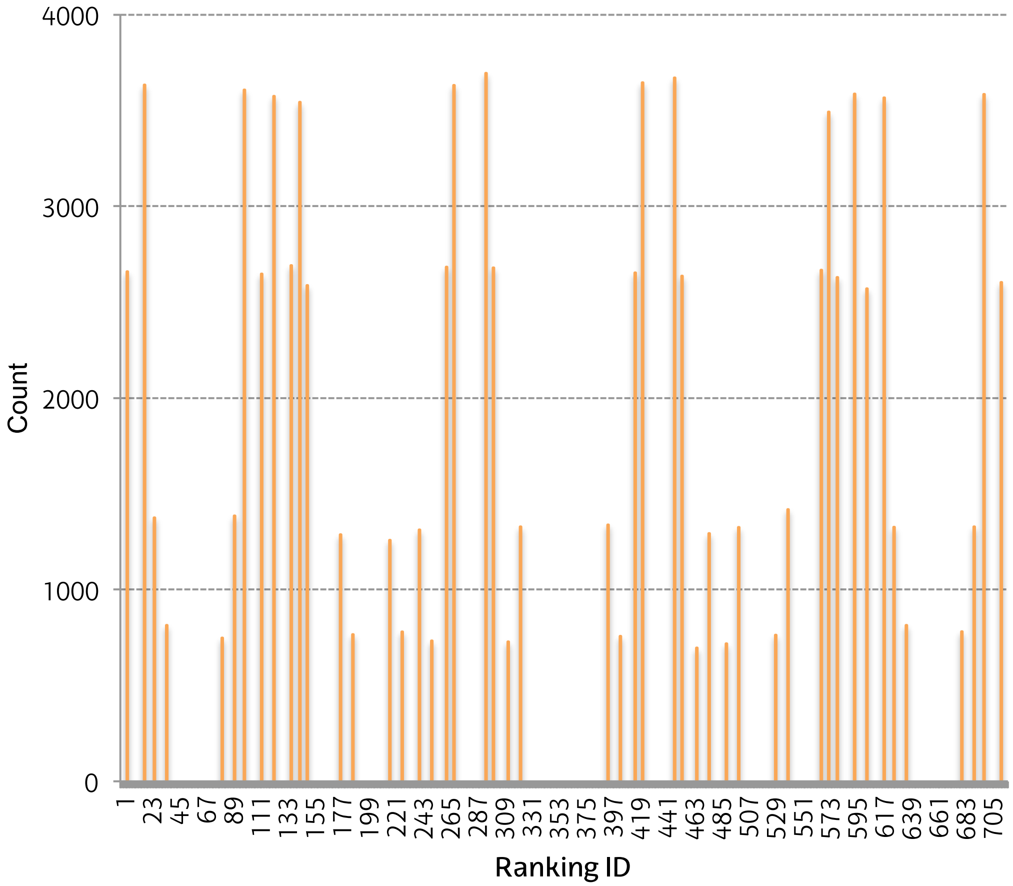
How many rankings can be generated?

$$(|\mathbb{S}_n|)! = (n!)!$$

$$(3!)! = 720$$

Are We Generating Instances Uniformly At Random?

10^5 LOP instances; n=3 → 720 possible rankings ; b_{kl} sampled from [0,100] u.a.r.

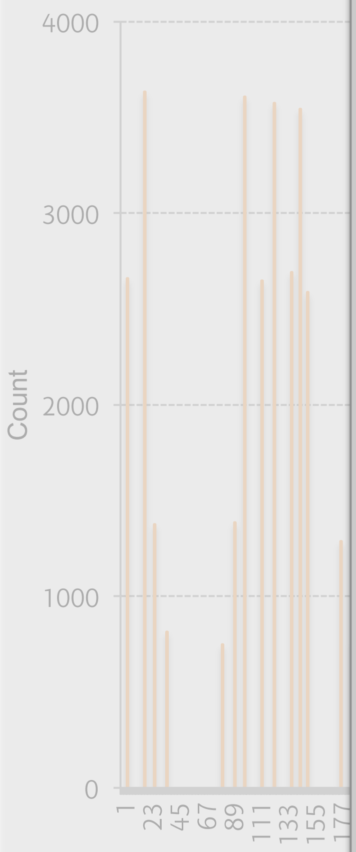


Only **48 different rankings**...

Are We Generating Instances Uniformly At Random?

10^5 LOP instar

rom [0,100] u.a.r.



8 different
nkings...

Linear Ordering Problem

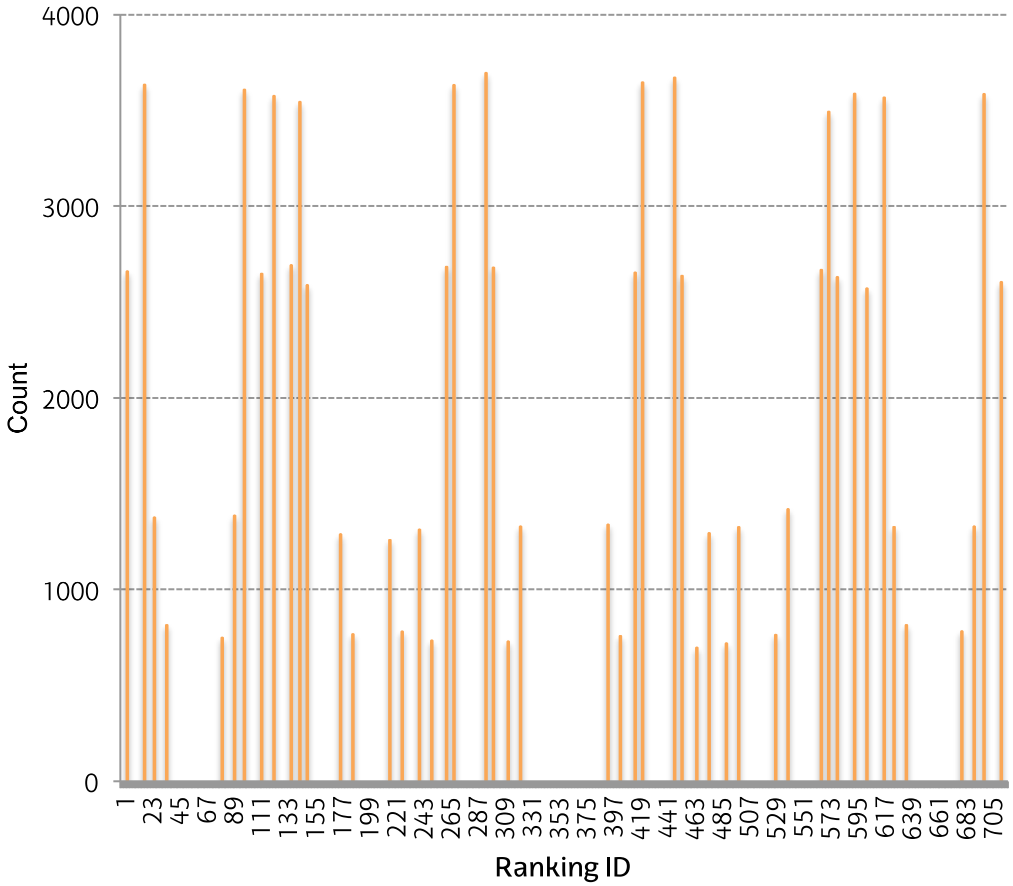
Reverse

$$\begin{aligned} \sigma_1 &= (14235) \\ \sigma_2 &= (32514) \\ &\vdots \\ \sigma_{n!-1} &= (41523) \\ \sigma_{n!} &= (53241) \end{aligned}$$

$$\left(\frac{n!}{2}\right)! \sum_{i=0}^{n!/2} \binom{n!/2}{i}$$

Are We Generating Instances Uniformly At Random?

10^5 LOP instances; ***n***=3 → 720 possible rankings ; $\boldsymbol{b}_{kl}$ sampled from [0,100] u.a.r.



Only **48 different rankings**…

The rankings were **not** uniformly sampled:

XL: 3560 ± 84
L: 2531 ± 62
M: 1268 ± 63
S: 752 ± 25

0	16	11
21	0	14
26	23	0

Experiment

Constraints among consecutive solutions

XL Ranking

$$\left.\begin{array}{l} 123 \rightarrow x_{12}+x_{13}+x_{23} \\ 132 \rightarrow x_{13}+x_{12}+x_{32} \\ 213 \rightarrow x_{21}+x_{23}+x_{13} \\ 312 \rightarrow x_{31}+x_{32}+x_{12} \\ 231 \rightarrow x_{23}+x_{21}+x_{31} \\ 321 \rightarrow x_{32}+x_{31}+x_{21} \end{array}\right\} \begin{array}{lcl} x_{12} & > & x_{21} \\ x_{23} & > & x_{32} \\ x_{12}+x_{32} & > & x_{21}+x_{23} \\ x_{21}+x_{23}+x_{13} & > & x_{31}+x_{32}+x_{12} \\ x_{32}+x_{12} & > & x_{23}+x_{21} \\ x_{23} & > & x_{32} \end{array}$$

Defined over all the parameters

L Ranking

$$\left.\begin{array}{l} 213 \rightarrow x_{13}+x_{21}+x_{23} \\ 123 \rightarrow x_{12}+x_{13}+x_{23} \\ 132 \rightarrow x_{13}+x_{12}+x_{32} \\ 231 \rightarrow x_{23}+x_{21}+x_{31} \\ 321 \rightarrow x_{32}+x_{31}+x_{21} \\ 312 \rightarrow x_{31}+x_{32}+x_{12} \end{array}\right\} \begin{array}{lcl} x_{21}+x_{23} & > & x_{12}+x_{32} \\ x_{21} & > & x_{12} \\ x_{23} & > & x_{32} \\ x_{13}+x_{12}+x_{32} & > & x_{23}+x_{21}+x_{31} \\ x_{23} & > & x_{32} \\ x_{21} & > & x_{12} \end{array}$$

M Ranking

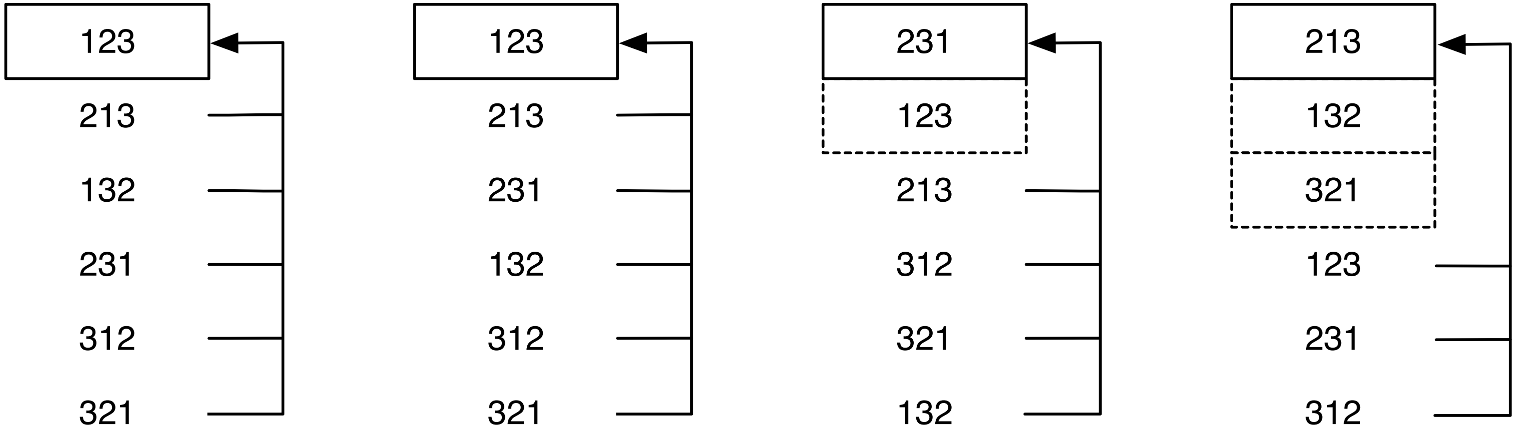
$$\left.\begin{array}{l} 321 \rightarrow x_{32}+x_{31}+x_{21} \\ 132 \rightarrow x_{13}+x_{12}+x_{32} \\ 312 \rightarrow x_{31}+x_{32}+x_{12} \\ 213 \rightarrow x_{21}+x_{23}+x_{13} \\ 231 \rightarrow x_{23}+x_{21}+x_{31} \\ 123 \rightarrow x_{12}+x_{13}+x_{23} \end{array}\right\} \begin{array}{lcl} x_{21} & > & x_{12} \\ x_{31}+x_{21} & > & x_{13}+x_{12} \\ x_{13} & > & x_{31} \\ x_{31}+x_{32}+x_{12} & > & x_{21}+x_{23}+x_{13} \\ x_{13} & > & x_{31} \\ x_{31}+x_{21} & > & x_{13}+x_{12} \end{array}$$

S Ranking

$$\left.\begin{array}{l} 213 \rightarrow x_{21}+x_{23}+x_{13} \\ 321 \rightarrow x_{32}+x_{31}+x_{21} \\ 132 \rightarrow x_{13}+x_{12}+x_{32} \\ 231 \rightarrow x_{23}+x_{21}+x_{31} \\ 123 \rightarrow x_{12}+x_{13}+x_{23} \\ 312 \rightarrow x_{31}+x_{32}+x_{12} \end{array}\right\} \begin{array}{lcl} x_{21}+x_{23} & > & x_{12}+x_{32} \\ x_{23}+x_{13} & > & x_{32}+x_{31} \\ x_{31}+x_{21} & > & x_{13}+x_{12} \\ x_{13}+x_{12}+x_{32} & > & x_{23}+x_{21}+x_{31} \\ x_{21}+x_{31} & > & x_{12}+x_{13} \\ x_{13}+x_{23} & > & x_{31}+x_{32} \end{array}$$

Are We Generating Instances Uniformly At Random?

Analysis of Local Optima

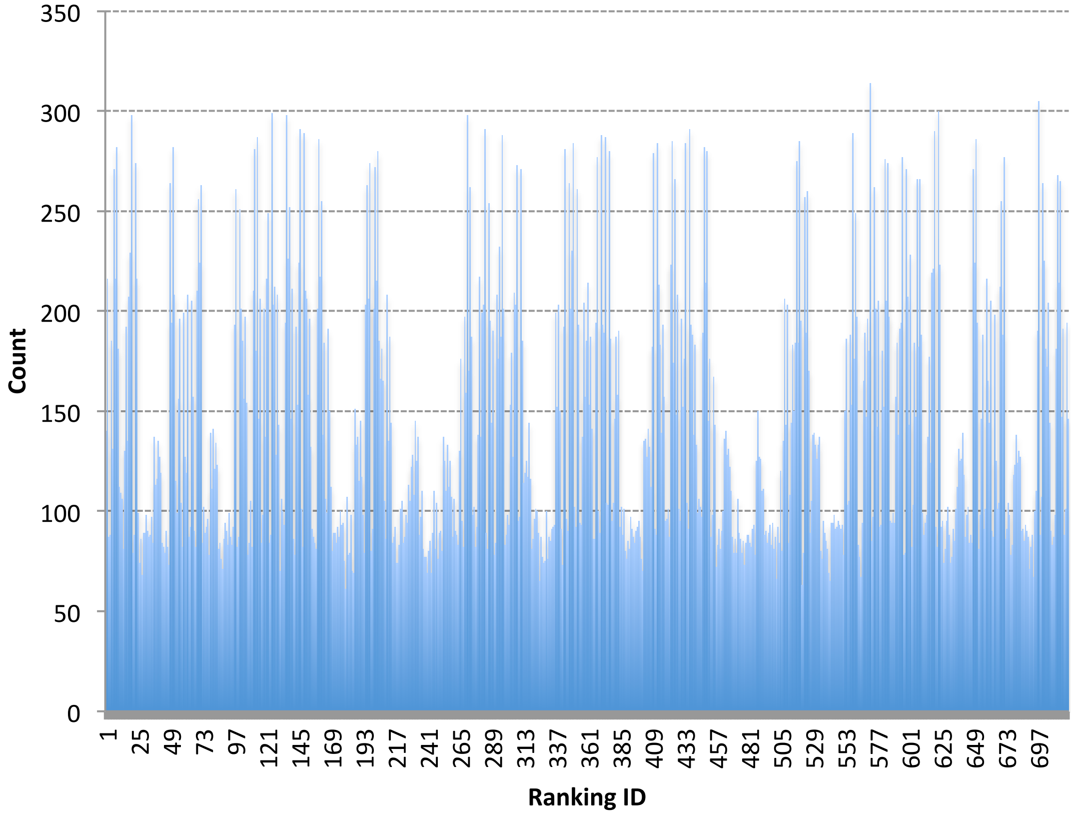


123	123	231	213
213	213	123	132
132	231	213	321
231	132	312	123
312	312	321	231
321	321	132	312
XL ranking	L ranking	M ranking	S ranking

Within each group equal number of local optima were found!
And at the same ranks!

Are We Generating Instances Uniformly At Random?

10^5 instances **QAP**; *n*=3 → 720 possible rankings ; parameters sampled from [0,100] u.a.r.

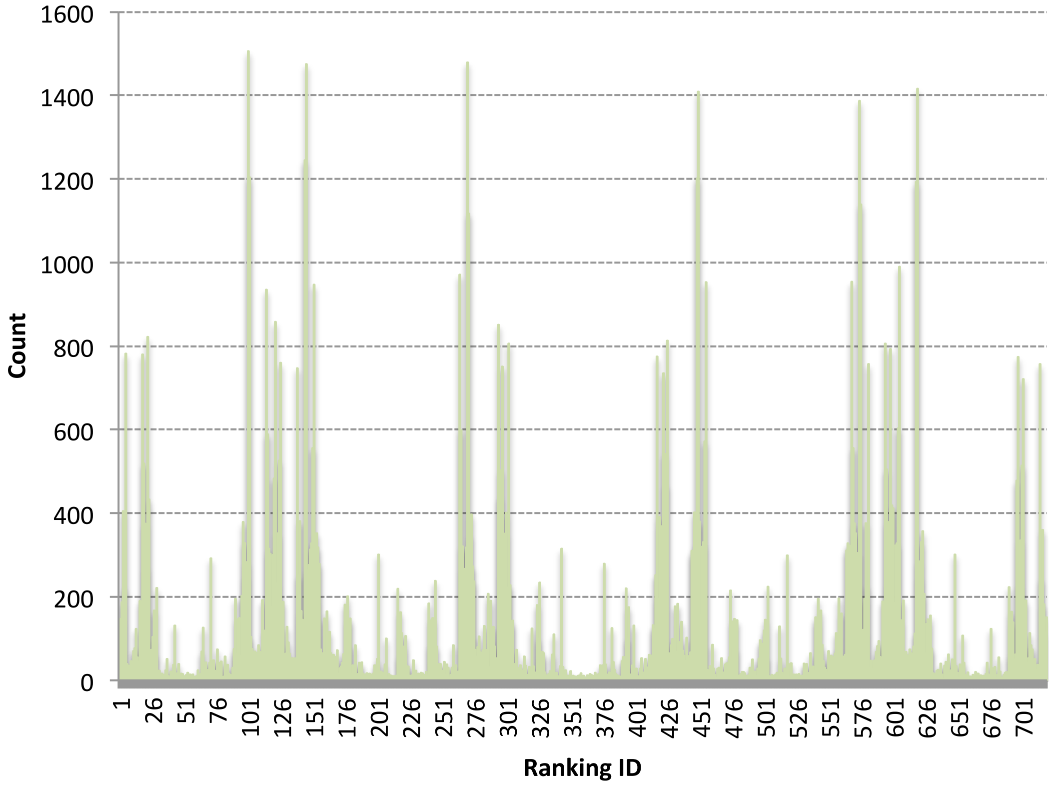


All the possible rankings were created (720)

Symmetry can be observed

Are We Generating Instances Uniformly At Random?

10^5 instances **PFSP**; n=3 x m=10 → 720 possible rankings ; parameters sampled from [0,100] u.a.r.



All the possible rankings were created (720)

Symmetry can be observed

Large variance

To take home…

1. Study the problem and gain insight
2. Algorithm design

3. A lot of research to do yet

Permutation-based Combinatorial Optimization Problems under the Microscope

Josu Ceberio

Thank you for your attention!

Intelligent Systems Group

eman ta zabal zazu

Universidad del País Vasco
Euskal Herriko Unibertsitatea